# A HYPERBOLIC METRIC AND STABILITY CONDITIONS ON K3 SURFACES WITH $\rho = 1$

KOTARO KAWATANI

## 1. Introduction

In this article we introduce a hyperbolic metric on the (normalized) space of stability conditions on projective K3 surfaces $X$ with Picard rank $\rho(X) = 1$. And we show that all walls are geodesic in the normalized space with respect to the hyperbolic metric. Furthermore we demonstrate how the hyperbolic metric is helpful for us by discussing mainly two topics. We first make a study of so called Bridgeland's conjecture. In the second topic we prove a famous Orlov's theorem without the global Torelli theorem.

Though Bridgeland's conjecture may be well-known, we would like to start from the review of it.

### 1.1. **Bridgeland's conjecture.**

In [Bri07] Bridgeland introduced the notion of *stability conditions* on arbitrary triangulated categories $\mathcal{D}$. By virtue of this we could define the notion of "$\sigma$-(semi)stability" for objects $E \in \mathcal{D}$ with respect to a stability condition $\sigma$ on $\mathcal{D}$.

Bridgeland also showed that each connected component of the space $\mathrm{Stab}(\mathcal{D})$ consisting of stability conditions on $\mathcal{D}$ is a complex manifold unless $\mathrm{Stab}(\mathcal{D})$ is empty. Hence the non-emptiness of $\mathrm{Stab}(\mathcal{D})$ is one of the biggest problem. Many researchers study this problem in various situations. For instance suppose $\mathcal{D}$ is the bounded derived category $D(M)$ of coherent sheaves on a projective manifold $M$. In the case of $\dim M = 1$, it was proven by [Oka06] (the genus is 0), [Bri07] (the genus is 1) and [Mac07] (the genus is greater than 1) and in the case of $\dim M = 2$, proven by [Bri08] (K3 or abelian surfaces) and [ABL07] (other surfaces). In the case of $\dim M = 3$ it is discussed by [BMT11]. We have to mention that these are just a handful of various studies.

As we stated before, the space $\mathrm{Stab}(X)$ of stability conditions on the derived category $D(X)$ of a projective K3 surface $X$ is not empty by [Bri08]. This fact is proven by finding a distinguished connected component $\mathrm{Stab}^{\dagger}(X)$. For $\mathrm{Stab}^{\dagger}(X)$ Bridgeland conjectured the following:

*Date*: March 2, 2013, version 3.

2010 *Mathematics Subject Classification.* Primary 14F05; Secondly 14J28, 18E30, 32Q45.

**Conjecture 1.1** (Bridgeland)**.** *The space* $\mathrm{Stab}(X)$ *is connected, that is,* $\mathrm{Stab}(X) = \mathrm{Stab}^{\dagger}(X)$*. Furthermore the distinguished component* $\mathrm{Stab}^{\dagger}(X)$ *is simply connected.*

As was proven by [Bri08], if the conjecture holds then we can determine the group structure of $\mathrm{Aut}(D(X))$ as follows. We have the covering map $\pi : \mathrm{Stab}^{\dagger}(X) \to \mathcal{P}_0^+(X)$ by [Bri08, Theorem 1.1] (See also Theorem 2.5). Here $\mathcal{P}_0^+(X)$ is a subset of $H^*(X, \mathbb{C})$ (See also Section 2.1). By virtue of [Bri08] and [HMS09], if Conjecture 1.1 holds we have the exact sequence of groups:

$$1 \to \pi_1(\mathcal{P}_0^+(X)) \to \mathrm{Aut}(D(X)) \overset{\rho}{\to} O^+_{\mathrm{Hodge}}(H^*(X, \mathbb{Z})) \to 1, \tag{1.1}$$

where $O^+_{\mathrm{Hodge}}(H^*(X, \mathbb{Z}))$ is the Hodge isometry group of $H^*(X, \mathbb{Z})$ which preserves the orientation of $H^*(X, \mathbb{Z})$. Hence Bridgeland's conjecture predicts that $\mathrm{Aut}(D(X))$ is given by an extension of $\pi_1(\mathcal{P}_0^+(X))$ and $O^+_{\mathrm{Hodge}}(H^*(X, \mathbb{Z}))$ and that the kernel $\mathrm{Ker}(\rho)$ of the representation $\rho$ is given by the fundamental group $\pi_1(\mathcal{P}_0^+(X))$.

### 1.2. **First theorem.**

Recall the right $\tilde{GL}^+(2, \mathbb{R})$-action on $\mathrm{Stab}(X)$ (See also Lemma A.6) where $\tilde{GL}^+(2, \mathbb{R})$ is the universal cover of $GL^+(2, \mathbb{R})$.

We define $\mathrm{Stab}^{\mathrm{n}}(X)$ by the quotient of $\mathrm{Stab}^{\dagger}(X)$ by the right $\tilde{GL}^+(2, \mathbb{R})$ action. We call it a *normalized stability manifold*. For a projective K3 surface with $\rho(X) = 1$, we first introduce a hyperbolic metric on $\mathrm{Stab}^{\mathrm{n}}(X)$. Then we show that the hyperbolic metric is independent of the choice of Fourier-Mukai partners of $X$ :

**Theorem 1.2** (=Theorem 3.3)**.** *Assume that* $\rho(X) = 1$*.*

(1) $\mathrm{Stab}^{\mathrm{n}}(X)$ *is a hyperbolic 2 dimensional manifold.*
(2) *Let* $Y$ *be a Fourier-Mukai partner of* $X$ *and* $\Phi : D(Y) \to D(X)$ *an equivalence which preserve the distinguished component* $\mathrm{Stab}^{\dagger}(X)$*. Then the induced morphism* $\Phi_*^{\mathrm{n}} : \mathrm{Stab}^{\mathrm{n}}(Y) \to \mathrm{Stab}^{\mathrm{n}}(X)$ *is an isometry with respect to the hyperbolic metric.*

Clearly if $\mathrm{Stab}(X)$ is connected it is unnecessary to assume that $\Phi$ preserves the distinguished component.

We remark that there is another study by Woolf which focus on the metric on $\mathrm{Stab}(\mathcal{D})$ (not normalized!). In [Woo12], he showed that $\mathrm{Stab}(\mathcal{D})$ is complete with respect to the original metric introduced by Bridgeland. Our study is the first work which focuses on a different structure from Bridgeland's original framework.

### 1.3. **Second theorem.**

Next, by using the hyperbolic structure, we observe the simply connectedness of $\mathrm{Stab}^{\dagger}(X)$ :

**Theorem 1.3** (=Theorem 4.1)**.** *Let* $X$ *be a projective K3 surface with* $\rho(X) = 1$*. The following three conditions are equivalent.*

(1) $\mathrm{Stab}^{\dagger}(X)$ *is simply connected.*
(2) $\mathrm{Stab}^{\mathrm{n}}(X)$ *is isomorphic to the upper half plain* $\mathbb{H}$.
(3) *The subgroup* $W(X)$[1] *of* $\mathrm{Aut}(D(X))$ *is isomorphic to the free group generated by* $T_A^2$ :

$$W(X) = \ast_A(\mathbb{Z} \cdot T_A^2),$$

*where $A$ runs through all spherical locally free sheaves and $\ast$ is the free product of groups generated by two times compositions of the spherical twist $T_A$ by $A$.*

We give two remarks on Theorem 4.1. Firstly we could not prove the simply connectedness. However by using the hyperbolic structure on $\mathrm{Stab}^{\mathrm{n}}(X)$, we can deduce the global geometry not only of $\mathrm{Stab}^{\mathrm{n}}(X)$ but also of $\mathrm{Stab}^{\dagger}(X)$. Since $\mathrm{Stab}^{\dagger}(X)$ is a $\tilde{GL}^+(2,\mathbb{R})$-bundle on $\mathrm{Stab}^{\mathrm{n}}(X)$, and we see $\mathrm{Stab}^{\dagger}(X)$ is simply connected if and only if it is a $\tilde{GL}^+(2,\mathbb{R})$-bundle over the upper half plain $\mathbb{H}$.

Secondly, if Conjecture 1.1 holds then we see the kernel $\mathrm{Ker}(\rho)$ is generated by $W(X)$ and the double shift [2]. Since the double shift [2] commutes with any equivalence, the freeness of $W(X)$ implies $\mathrm{Ker}(\rho)/\mathbb{Z}[2]$ is free. However in the higher Picard rank cases, it is thought that the generators of $\mathrm{Ker}(\rho)/\mathbb{Z}[2]$ have relations (See also Remark 4.3). Hence the freeness of $W(X)$ is a special phenomena.

1.4. **Third theorem.**

In the third theorem, we study chamber structures on $\mathrm{Stab}^{\dagger}(X)$ in terms of the hyperbolic structure on $\mathrm{Stab}^{\mathrm{n}}(X)$. Before we state the third theorem, let us recall chamber structures.

For a set $\mathcal{S} \subset D(X)$ of objects which has bounded mass (See Definition B.1) and an arbitrary compact subset $B \subset \mathrm{Stab}^{\dagger}(X)$, we can define a finite collection of real codimension 1 submanifolds $\{W_\gamma\}_{\gamma\in\Gamma}$ satisfying the following property:

- Let $C \subset B \setminus \bigcup_{\gamma\in\Gamma} W_\gamma$ be an arbitrary connected component. If $E \in \mathcal{S}$ is $\sigma$-semistable for some $\sigma \in C$ then $E$ is $\tau$-semistable for all $\tau \in C$.

Each $W_\gamma$ is said to be a *wall* and each connected component $C$ is said to be a *chamber*. We call all data of chambers and walls a *chamber structure*. We remark that chamber structures on $\mathrm{Stab}^{\dagger}(X)$ descend to the normalized stability manifold $\mathrm{Stab}^{\mathrm{n}}(X)$. Namely $C/\tilde{GL}^+(2,\mathbb{R})$ and $\{W_\gamma/\tilde{GL}^+(2,\mathbb{R})\}$ also define a chamber structure on $\mathrm{Stab}^{\mathrm{n}}(X)$. Our third theorem is the following:

**Theorem 1.4** (=Theorem 5.5)**.** *All walls of chamber structures of* $\mathrm{Stab}^{\mathrm{n}}(X)$ *are geodesic.*

[1]The definition of $W(X)$ appears in Definition 2.3.

1.5. **Revisit of Orlov's theorem.**

Generally speaking Fourier-Mukai transformations on $X$ may change chamber structures. By Theorems 3.3 and 5.5, we see that the image of walls by Fourier-Mukai transformations is also geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$. Applying this observation we show the following:

**Proposition 1.5** (=Proposition 6.4)**.** *Let $X$ be a projective K3 surface with $\rho(X) = 1$ and $Y$ a Fourier-Mukai partner of $X$ with an equivalence $\Phi : D(Y) \to D(X)$.*

*If the induced morphism $\Phi_* : \mathrm{Stab}(Y) \to \mathrm{Stab}(X)$ preserves the distinguished component, then $Y$ is isomorphic to the fine moduli space of Gieseker stable torsion free sheaves.*

We have to mention that a more stronger statement was already proven by Orlov in [Orl97]; Any Fourier-Mukai partner of projective K3 surfaces is isomorphic to the fine moduli space of Gieseker stable sheaves. Our proof never needs the global Torelli theorem but Orlov's proof needs it. Hence our proof gives a new feature of stability conditions.

1.6. **A Rough sketch of the proof of Proposition 6.4.**

Let us recall a chamber $U(X) \subset \mathrm{Stab}^{\dagger}(X)$ (roughly) consisting of $\sigma$ such that any structure sheaves $\mathcal{O}_x$ of closed points $x \in X$ are $\sigma$-stable (See also Section 2.2). By using $U(X)$ and the group $W(X) \subset \mathrm{Aut}(D(X))$, the distinguished component $\mathrm{Stab}^{\dagger}(X)$ is written as $\mathrm{Stab}^{\dagger}(X) = \bigcup_{\Psi \in W(X)} \Psi_* \bar{U}(X)$ where $\bar{U}(X)$ is the closure of $U(X)$. The boundary $\bar{U}(X) \setminus U(X)$ is denoted by $\partial U(X)$. We note that the chamber $U(X)$ depends on the choice of Fourier-Mukai partners.

Now we give a rough sketch of the proof of Proposition 6.4. By the assumption we first see that there is an equivalence $\Psi \in W(X)$ such that $\Phi_*(U(Y)) \cap \Psi_*^{-1}(U(X)) \neq \emptyset$ and also $\Psi_* \circ \Psi_*(U(Y)) \cap U(X) \neq \emptyset$. Then it is enough to show that $\Psi \circ \Phi(\mathcal{O}_y)$ is a Gieseker stable torsion free sheaf by using Bridgeland's large volume limit argument [Bri08, Proposition 14.2]. When we apply the large volume limit argument we have to see where the boundary $\Psi_* \circ \Phi_*(\partial U(Y))$ appears in $U(X)$. Since $\Psi_* \circ \Phi_*(\partial U(Y))/\tilde{GL}^+(2, \mathbb{R})$ is geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$ , we can explicitly determine $\Psi_* \circ \Phi_*(\partial U(Y)) \cap U(X)$. This is why we can prove Proposition 6.4.

1.7. **Contents.**

Finally we explain the contents of this article. In this article there are two appendixes, where we survey Bridgeland's theory. In Section 2 we prepare some basic terminologies. In Section 3 we prove the first main theorem. In Section 4 we prove the second main theorem. The third theorem will be proven in Section 5. The analysis of $\partial U(X)$, which is necessary for Theorem 4.1, will be also done in Section 5. In Section 6 we revisit Orlov's theorem.

1.8. **Acknowledgement.** The author thanks Professor Y. Namikawa for the comments which improve the readability of introduction.

## 2. Preliminaries

In this section we prepare basic notations and lemmas.

2.1. **Terminologies.** Let $X$ be a projective K3 surface. The abelian category of coherent sheaves on $X$ is denoted by $\mathrm{Coh}(X)$. Note that the numerical Grothendieck group $\mathcal{N}(X)$ is isomorphic to $H^0(X,\mathbb{Z})\oplus \mathrm{NS}(X)\oplus H^4(X,\mathbb{Z})$. We put $v(E)=ch(E)\sqrt{td_X}$ for $E\in D(X)$. Then we see

$$v(E)=r_E\oplus c_E\oplus s_E\in\mathcal{N}(X).$$

One can easily check that $r_E=\operatorname{rank}E$, $c_E$ is the first Chern class $c_1(E)$ and $s_E=\chi(X,E)-\operatorname{rank}E$. Hence for a vector $v=r\oplus c\oplus s\in\mathcal{N}(X)$, the component $r$ is called the *rank* of $v$.

The Mukai pairing $\langle,\rangle$ on $H^*(X,\mathbb{Z})$ is given by

$$\langle r\oplus c\oplus s, r'\oplus c'\oplus s'\rangle = cc'-rs'-r's.$$

By Riemann-Roch theorem we see

$$\chi(E,F)=\sum_i(-1)^i\dim\mathrm{Hom}^i_{D(X)}(E,F)=-\langle v(E),v(F)\rangle.$$

An object $A\in D(X)$ is said to be *spherical* if $A$ staisfies

$$\mathrm{Hom}^i_{D(X)}(A,A)=\begin{cases}\mathbb{C} & (i=0,2)\\ 0 & (\text{otherwise}).\end{cases}$$

We note that $v(A)^2=-2$ if $A$ is spherical. By the effort of [ST01], for a spherical object $A$ we could define the autoequivalence $T_A$ called a *spherical twist* (See also [Huy07, Chapter 8]). By the definition of $T_A$ we have the following distinguished triangle for $E\in D(X)$:

$$\mathrm{Hom}^*_{D(X)}(A,E)\otimes A\xrightarrow{\mathrm{ev}}E\longrightarrow T_A(E), \tag{2.1}$$

where ev is the evaluation map. We call the above triangle a *spherical triangle.* We note that the vector of $T_A(E)$ can be calculated as follows

$$v(T_A(E))=v(E)+\langle v(E),v(A)\rangle v(A).$$

Let $\mathcal{C}(X)$ be the set $\{v\in\mathrm{NS}(X)_{\mathbb{R}}|v^2>0\}$ and $\mathcal{C}(X)^+$ the connected component containing an ample divisor $\omega$, where $\mathrm{NS}(X)_{\mathbb{R}}=\mathrm{NS}(X)\otimes\mathbb{R}$. Let $\Delta(X)$ be the set of $(-2)$-vectors:

$$\Delta(X)=\{\delta\in\mathcal{N}(X)|\delta^2=-2\}$$

and let $\Delta^+(X)$ be the set $\{\delta\in\Delta(X)|\delta=r\oplus c\oplus s, r>0\}$.

Following [Bri08] and [Har10], we put

$$\mathcal{P}(X)=\{v\in\mathcal{N}(X)\otimes\mathbb{C}|\mathfrak{Re}(v)\text{ and }\mathfrak{Im}(v)\text{ span a positive 2 plain}\}$$

and also

$$\mathfrak{D}(X)=\{[v]\in\mathbb{P}(\mathcal{N}(X)\otimes\mathbb{C})|v^2=0, v\bar{v}>0\}.$$

Since $\mathcal{P}(X)$ and $\mathfrak{D}(X)$ have two connected components, we define $\mathcal{P}^+(X)$ (respectively $\mathfrak{D}^+(X)$) by the connected component containing $\exp(\sqrt{-1}\omega)$

(respectively $[\exp(\sqrt{-1}\omega)]$). Note that $\mathfrak{D}^+(X)$ is isomorphic to $\mathrm{NS}(X)_{\mathbb{R}} \times \mathcal{C}^+(X)$ via the following map:

$$\mathrm{NS}(X)_{\mathbb{R}} \times \mathcal{C}^+(X) \ni (\beta, \gamma) \mapsto [\exp(\beta + \sqrt{-1}\gamma)] \in \mathfrak{D}^+(X).$$

As is shown in [Har10] (and also written in [Bri08]), $\mathcal{P}^+(X)$ is a principle $GL^+(2,\mathbb{R})$-bundle which has a section given by the exponential map. Thus $\mathcal{P}^+(X)$ is isomorphic to $\mathfrak{D}^+(X) \times GL^+(2,\mathbb{R})$. We put $\mathcal{P}^+_0(X)$ by

$$\mathcal{P}^+_0(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^{\perp}$$

where $\langle \delta \rangle^{\perp}$ is the orthogonal complement of $\delta$ with respect to the Mukai pairing on $H^*(X,\mathbb{Z})$. We also define

$$\mathfrak{D}^+_0(X) = \{[v] \in \mathfrak{D}^+(X) | \langle v, \delta \rangle \neq 0 \ (\forall \delta \in \Delta(X))\}.$$

Then we see $\mathcal{P}^+_0(X)$ is isomorphic to $\mathfrak{D}^+_0(X) \times GL^+(2,\mathbb{R})$.

2.2. **Stability conditions on K3 surfaces.** A short survey of stability conditions introduced by [Bri07] is given in Appendix A.

Let $\mathrm{Stab}(X)$ be the set of numerical locally finite stability conditions on $D(X)$. We put $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(X)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ and $Z$ is a central charge. Since the Mukai paring is non-degenerate on $\mathcal{N}(X)$ we have the natural map:

$$\pi : \mathrm{Stab}(X) \to \mathcal{N}(X) \otimes \mathbb{C}, \ \pi(\sigma) = Z^{\vee}$$

where $Z(E) = \langle Z^{\vee}, v(E) \rangle$.

In $\mathrm{Stab}(X)$, there is a connected component $\mathrm{Stab}^{\dagger}(X)$ which contains the set $U(X)$ :

$$U(X) = \{\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(X) | Z^{\vee} \in \mathcal{P}(X) \setminus \cup_{\delta \in \Delta(X)} \langle \delta \rangle^{\perp}, \\ \mathcal{O}_x \text{ is } \sigma\text{-stable in the same phase for all } x \in X\}.$$

Let $U(X)$ be the closure of $U(X)$ in $\mathrm{Stab}(X)$. Then we see that $\bar{U}(X)$ be the set of stability conditions $\sigma$ such that $\mathcal{O}_x$ ($\forall x \in X$) is $\sigma$-semistable in the same phase with $Z^{\vee} \in \mathcal{P}(X) \setminus \cup_{\delta \in \Delta(X)} \langle \delta \rangle^{\perp}$. We define $\partial U(X)$ by $\bar{U}(X) \setminus U(X)$ and call it the *boundary of* $U(X)$. A brief survey for the boundary $\partial U(X)$ is given in Appendix B.

We define the set $V(X)$ by

$$V(X) = \{\sigma = (\mathcal{A}, Z) \in U(X) | Z(\mathcal{O}_x) = -1, \ \mathcal{O}_x \text{ is } \sigma\text{-stable with phase } 1\}.$$

One can see $U(X) = V(X) \cdot \tilde{GL}^+(2,\mathbb{R}) \cong V(X) \times \tilde{GL}^+(2,\mathbb{R})$ by [Bri08, Proposition 10.3]. Furthermore the set $V(X)$ is parametrized by $(\beta, \omega) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)$ in the following way:

Let $(\beta, \omega) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)$. For such a pair $(\beta, \omega)$, we define $\mathcal{A}_{(\beta,\omega)}$ and $Z_{(\beta,\omega)}$ as follows :

$$\begin{aligned} \mathcal{A}_{(\beta,\omega)} &:= \left\{E^\bullet \in D(X) \middle| H^i(E^\bullet) \begin{cases} \in \mathcal{T}_{(\beta,\omega)} & (i=0) \\ \in \mathcal{F}_{(\beta,\omega)} & (i=-1) \\ =0 & (\text{otherwise}) \end{cases} \right\} \\ Z_{(\beta,\omega)}(E) &:= \langle \exp(\beta + \sqrt{-1}\omega), v(E)\rangle, \end{aligned}$$

where

$$\begin{aligned} \mathcal{T}_{(\beta,\omega)} &:= \{E \in \mathrm{Coh}(X) | E \text{ is a torsion sheaf or } \mu^-_\omega(E/\text{torsion}) > \beta\omega\} \text{ and} \\ \mathcal{F}_{(\beta,\omega)} &:= \{E \in \mathrm{Coh}(X) | E \text{ is torsion free and } \mu^+_\omega(E) \le \beta\omega\}. \end{aligned}$$

Here $\mu^+_\omega(E)$ (respectively $\mu^-_\omega(E)$) is the maximal slope (respectively minimal slope) of semistable factors of a torsion free sheaf $E$ with respect to the slope stability. Since the pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$ gives a torsion pair on $\mathrm{Coh}(X)$, $\mathcal{A}_{(\beta,\omega)}$ is the heart of a bounded $t$-structure on $D(X)$. We denote the pair $(\mathcal{A}_{(\beta,\omega)}, Z_{(\beta,\omega)})$ by $\sigma_{(\beta,\omega)}$.

**Proposition 2.1** ([Bri08, Proposition 10.3])**.** *Assume that $(\beta,\omega)$ satisfies the condition*

$$\langle \exp(\beta + \sqrt{-1}\omega), \delta\rangle \notin \mathbb{R}_{\le 0}, (\forall \delta \in \Delta^+(X)) \tag{2.2}$$

*Then the pair $\sigma_{(\beta,\omega)}$ gives a numerical locally finite stability condition on $D(X)$. Furthermore we have*

$$V(X) = \{\sigma_{(\beta,\omega)} \in \mathrm{Stab}^\dagger(X) | (\beta,\omega) \textit{ satisfies the condition (2.2)}\}.$$

**Remark 2.2.** We put $v(E) = r_E \oplus c_1(E) \oplus s_E$ for $E \in D(X)$ . As the author remarked in [Kaw10, Section 4, (4.1)], for objects $E \in D(X)$ with $\mathrm{rank}\, E \neq 0$, we can rewrite $Z_{(\beta,\omega)}(E)$ as follows,

$$Z_{(\beta,\omega)}(E) = \frac{v(E)^2}{2r_E} + \frac{r_E}{2}\Big(\omega + \sqrt{-1}\big(\frac{c_1(E)}{r_E} - \beta\big)\Big)^2. \tag{2.3}$$

This equation (2.3) plays an important role in Lemma 3.2 which is crucial for Theorem 3.3.

**Definition 2.3.** For a projective K3 surface we define a subgroup $W(X)$ of $\mathrm{Aut}(D(X))$ generated by

$$W(X) = \langle T_A^2, T_{\mathcal{O}_C(k)} | k \in \mathbb{Z}, A, C\rangle$$

where $A$ runs all spherical locally free sheaves and $C$ runs all $(-2)$-curves in $X$. In particular if $\rho(X) = 1$ then $W(X)$ is generated by $T_A^2$.

Then by using $U(X)$ and $W(X)$ we can describe $\mathrm{Stab}^\dagger(X)$ in a explicit way:

**Proposition 2.4** ([Bri08, Proposition 13.2])**.** *The distinguished connected component* $\mathrm{Stab}^{\dagger}(X)$ *is given by*

$$\mathrm{Stab}^{\dagger}(X) = \bigcup_{\Phi \in W(X)} \Phi_* \bar{U}(X).$$

**Theorem 2.5** ([Bri08])**.** *The natural map* $\pi : \mathrm{Stab}^{\dagger}(X) \to \mathcal{N}(X) \otimes \mathbb{C}$ *is surjective to* $\mathcal{P}_0^+(X)$*. Furthermore* $\pi$ *is a Galois covering. The covering transformation group is the subgroup generated by equivalences in* $\mathrm{Ker}(\rho)$ *which preserves* $\mathrm{Stab}^{\dagger}(X)$*.*

**Corollary 2.6.** *For a projective K3 surface, the induced map*

$$\pi^{\mathrm{n}} : \mathrm{Stab}^{\mathrm{n}}(X) \to \mathfrak{D}_0^+(X)$$

*is also a Galois covering map.*

*Proof.* We have the following $GL^+(2,\mathbb{R})$-equivariant diagram:

$$\begin{array}{ccc} \mathrm{Stab}^{\dagger}(X)/\mathbb{Z}[2] & \xrightarrow{\pi'} & \mathcal{P}_0^+(X) \\ \downarrow & & \downarrow \\ \mathrm{Stab}^{\mathrm{n}}(X) & \xrightarrow{\pi^{\mathrm{n}}} & \mathfrak{D}_0^+(X). \end{array}$$

We note that both vertical maps are $GL^+(2,\mathbb{R})$-bundles and that $\pi'$ is also a Galois covering.

By Theorem 2.5 the covering transformation group of $\pi'$ is a subgroup of $\mathrm{Aut}(D(X))/\mathbb{Z}[2]$. Hence the right $GL^+(2,\mathbb{R})$-action on $\mathrm{Stab}^{\dagger}(X)/\mathbb{Z}[2]$ commutes with the covering transformations. Hence $\pi^{\mathrm{n}}$ is also a Galois covering. □

2.3. **On the fundamental group of $\mathcal{P}_0^+(X)$.** We are interested in the fundamental group $\pi_1(\mathcal{P}_0^+(X))$. Generally speaking, it is highly difficult to describe the above condition (2.2) explicitly. Because of this difficulty, it becomes difficult to determine the relation between generators of $\pi_1(\mathcal{P}_0^+(X))$. Hence it seems impossible to determine the group structure of $\pi_1(\mathcal{P}_0^+(X))$. However, under the assumption $\mathrm{NS}(X) = \mathbb{Z}L$ it becomes easier. Thus, in the following, we always assume $\mathrm{NS}(X) = \mathbb{Z}L$. We define $(X, L)$ by a pair with $\mathrm{NS}(X) = \mathbb{Z}L$.

For $(X, L)$, we define $\mathfrak{H}(X)$ by $\mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)$. Clearly there is a canonical isomorphism

$$\mathfrak{H}(X) \ni (\beta, \omega) = (xL, yL) \mapsto x + \sqrt{-1}y \in \mathbb{H},$$

where $\mathbb{H}$ is the upper half plain. Furthermore we put $\bar{\mathfrak{H}}(X)$ by

$$\bar{\mathfrak{H}}(X) = \{(xL, yL) | x \in \mathbb{R}, y \in \mathbb{R}_{\geq 0}\} \cup \{\infty\}.$$

We usually identify $\mathfrak{H}(X)$ (respectively $\bar{\mathfrak{H}}(X)$) with $\mathbb{H}$ (respectively $\bar{\mathbb{H}}$) by the above morphism. We also denote an element $(\beta, \omega) \in \bar{\mathfrak{H}}(X)$ by $\beta + \sqrt{-1}\omega$.

As we remarked before, since we have $\mathrm{Amp}(X) = \mathcal{C}^+(X)$ we see $\mathfrak{H}(X)$ is isomorphic to the symmetric domain $\mathfrak{D}^+(X)$ by the map:

$$\mathfrak{H}(X) \ni (\beta, \omega) \mapsto [\exp(\beta + \sqrt{-1}\omega)] \in \mathfrak{D}^+(X).$$

**Definition 2.7.** Let $\delta = r \oplus c \oplus s \in \Delta(X)$. An associated point $p \in \mathfrak{H}(X)$ with $\delta \in \Delta(X)$ is the point $p \in \mathfrak{H}(X)$ such that $\langle \exp(p), \delta \rangle = 0$. We also denote the point by $p(\delta)$ and call it a *spherical point*. If $\delta$ is the Mukai vector of a spherical object $A$ we denote simply $p(v(A))$ by $p(A)$.

**Remark 2.8.** Let $\delta \in \Delta(X)$ and we put $\delta = r \oplus c \oplus s$. Since $c^2 \geq 0$ we see $r \neq 0$. Thus we have the disjoint sum $\Delta(X) = \Delta^+(X) \sqcup (-\Delta^+(X))$.

Now we have the explicit description of $p(\delta)$ as follows:

$$p(\delta) = (\frac{c}{r}, \frac{1}{\sqrt{d}|r|}L) \in \mathfrak{H}(X),$$

where we put $L^2 = 2d$. Moreover one sees $p(\delta) = p(-\delta)$. Thus one can easily see that the following map is an isomorphism:

$$\mathfrak{H}_0(X) \ni (\beta, \omega) \mapsto [\exp(\beta + \sqrt{-1}\omega)] \in \mathfrak{D}_0^+(X),$$

where

$$\mathfrak{H}_0(X) = \mathfrak{H}(X) \setminus \{p(\delta) \in \mathfrak{H}(X) | \delta \in \Delta^+(X)\}.$$

The key lemma of this subsection is that the set $\{p(\delta) \in \mathfrak{H}(X) | \delta \in \Delta(X)\}$ is discreet in $\mathfrak{H}(X)$. To show this claim we introduce some notations.

**Definition 2.9.** Let $\delta = r \oplus c \oplus s \in \Delta^+(X)$.

1. We define the set $\Delta^{(i)}(X)$ by

$$\Delta^{(i)}(X) = \{r \oplus c \oplus s \in \Delta^+(X) | r \text{ is the } i\text{-th smallest in } \Delta^+(X)\}.$$

We also define the rank of $\Delta^{(i)}(X)$ by $r$ for some $\delta = r \oplus c \oplus s \in \Delta^{(i)}(X)$.

2. We define the subset $\mathcal{V}(X)$ of $\mathfrak{H}(X)$ as follows.

$$\mathcal{V}(X) = \{(\beta, \omega) \in \mathfrak{H}(X) | (\beta, \omega) \text{ satisfies the condition (2.2)}\}.$$

As we remarked in Proposition 2.1 this set is isomorphic to $V(X)$ consisting of stability conditions by the natural morphism $\pi$.

3. Let $r_i$ be the rank of $\Delta^{(i)}(X)$. We define the subset $\mathcal{V}^{(i)}(X)$ of $\mathcal{V}(X)$ by

$$\mathcal{V}^{(i)}(X) = \{(\beta, \omega) \in \mathcal{V}(X) | \omega^2 > \frac{2}{r_i^2}\}.$$

**Remark 2.10.** Let $X$ be a projective (not necessary Picard rank one) K3 surface. For any $\delta = r \oplus c \oplus s \in \Delta(X)$ with $r \geq 0$, there exists a spherical sheaf $A$ on $X$ such that $v(A) = \delta$ by [Kul90]. In particular if $r > 0$ then we can take $A$ as a locally free sheaf. In addition if we assume $\mathrm{NS}(X) = \mathbb{Z}L$ then we see $A$ is Gieseker-stable by [Muk87, Proposition 3.14]. Since we see $\gcd(r, n) = 1$ where $n$ satisfies $nL = c$, $A$ is $\mu$-stable by [HL97, Lemma 1.2.14].

**Remark 2.11.** For instance $\Delta^{(1)}(X)$ is the set of Mukai vectors of line bundles on $X$. Thus $\mathrm{rank}\,\Delta^{(1)}(X) = 1$ for any $(X, L)$. However for $i > 1$, the rank of $\Delta^{(i)}(X)$ depends on the degree $L^2$.

Since $\mathrm{rank}\,\Delta^{(1)}(X) = 1$, we see $(\beta, \omega)$ is in $\mathcal{V}^{(1)}(X)$ if and only if $\omega^2 > 2$. We have the following infinite filtration of $\mathcal{V}^{(i)}(X)$ $(i = 1, 2, 3 \cdots)$

$$\mathcal{V}^{(1)}(X) \subset \mathcal{V}^{(2)}(X) \subset \cdots \subset \mathcal{V}^{(n)}(X) \subset \cdots \subset \mathcal{V}(X).$$

**Lemma 2.12.** *Notations being as above,*

(1) *the set* $\mathfrak{S} = \{p(\delta) \in \mathfrak{H}(X) | \delta \in \Delta(X)\}$ *is a discreet set in* $\mathfrak{H}(X)$.
(2) *Furthermore the set* $\mathcal{V}(X)$ *is open in* $\mathfrak{H}(X)$.

*Proof.* Suppose that $\mathrm{NS}(X) = \mathbb{Z}L$ with $L^2 = 2d$. Let $p(\delta)$ be the spherical point of $\delta \in \Delta^+(X)$. We put $\delta = r \oplus c \oplus s$ where $c = nL$ for some $n \in \mathbb{Z}$.

Recall that $p(\delta)$ is given by

$$p(\delta) = (\frac{nL}{r}, \frac{1}{\sqrt{d}r}L).$$

We also note that $\gcd(r, n) = 1$ since $\delta^2 = -2$ and $\mathrm{NS}(X) = \mathbb{Z}L$. Let $B_\epsilon$ be the open ball whose center is $p(\delta)$ and the radius is $\epsilon$ (with respect to the usual metric). Since $r_{i+1} \geq r_i + 1$ (where $r_i$ is the rank of $\Delta^{(i)}(X)$) if $\epsilon$ is smaller than $\frac{1}{\sqrt{d}}(\frac{1}{r} - \frac{1}{r+1})$ we see $B_\epsilon \cap \mathfrak{S} = \{p(\delta)\}$.

We prove the second assertion. We define $S(\delta)$ for $\delta \in \Delta^+(X)$ as follows:

$$S(\delta) = \{(\beta, \omega) \in \mathfrak{H}(X) | \beta = \frac{c}{r}, 0 < \omega^2 \leq \frac{2}{r^2}\}.$$

Then one can check that

$$\mathcal{V}(X) = \mathfrak{H}(X) \setminus \bigcup_{\delta \in \Delta^+(X)} S(\delta).$$

Hence we see

$$\mathcal{V}^{(i)}(X) = \{(\beta, \omega) \in \mathfrak{H}(X) | \omega^2 > \frac{2}{r_i^2}\} \setminus \bigcup_{\delta \in \Delta^{(\leq i-1)}} S(\delta),$$

where $\Delta^{(\leq i)} = \bigcup_{j=1}^{i} \Delta^{(j)}(X)$. Since the set

$$\{\frac{c}{r} | \delta = r \oplus c \oplus s \in \Delta^{(\leq i)}\}$$

is discreet in $\mathbb{R}L$, the set $\mathcal{V}^{(i)}(X)$ is open in $\mathfrak{H}(X)$. Since we have

$$\mathcal{V}(X) = \bigcup_{i \in \mathbb{N}} \mathcal{V}^{(i)}(X),$$

the set $\mathcal{V}(X)$ is open in $\mathfrak{H}(X)$. □

**Definition 2.13.** We set elements of the fundamental groups $\pi_1(\mathfrak{H}_0(X))$ and of $\pi_1(GL^+(2, \mathbb{R}))$ as follows.

- We define $\ell_\delta$ by the loop which turns round only the point $p(\delta) \in \mathfrak{H}(X)$ counterclockwise;

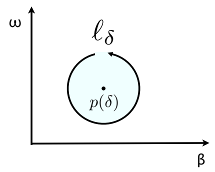


FIGURE 1. For $p(\delta)$ we define the loop $\ell_\delta$ as the above direction. We also assume that there are no spherical points $p(\delta')$ in the inside of $\ell_\delta$ except for $p(\delta)$ itself.

- We define $g \in \pi_1(GL^+(2,\mathbb{R}))$ by
$$g : [0,1] \ni t \mapsto \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \in GL^+(2,\mathbb{R}).$$

We note that $g$ is a generator of $\pi_1(GL^+(2,\mathbb{R}))$ since $\pi_1(GL^+(2,\mathbb{R})) \cong \pi_1(SO(2)) \cong \mathbb{Z}$.

**Proposition 2.14.** *The fundamental group $\pi_1(\mathcal{P}_0^+(X))$ is isomorphic to*
$$(\ast_{\delta \in \Delta^+(X)} \mathbb{Z} \cdot \ell_\delta) \times \mathbb{Z} \cdot g$$
*where $\ast_{\delta \in \Delta^+} \mathbb{Z} \cdot \ell_\delta$ is a free product of infinite cyclic groups $\mathbb{Z}$ generated by $\ell_\delta$.*

*Proof.* Since $\mathcal{P}_0^+(X)$ is isomorphic to $\mathfrak{D}_0^+(X) \times GL^+(2,\mathbb{R})$ we see $\pi_1(\mathcal{P}_0^+(X)) \cong \pi_1(\mathfrak{D}_0^+(X)) \times \mathbb{Z} \cdot g$. As we remarked before we have $\Delta(X) = \Delta^+(X) \sqcup (-\Delta^+(X))$. Hence we see
$$\mathfrak{D}_0^+(X) = \mathfrak{D}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^\perp = \mathfrak{D}^+(X) \setminus \bigcup_{\delta \in \Delta^+(X)} \langle \delta \rangle^\perp$$

Since $\mathfrak{D}_0^+(X)$ is isomorphic to $\mathfrak{H}_0(X)$ it is enough to show that
$$\pi_1(\mathfrak{H}_0(X)) = \ast_{\delta \in \Delta^+} \mathbb{Z} \cdot \ell_\delta$$

We choose a base point of $\mathfrak{H}_0(X)$ so that $p = \sqrt{-1}\omega$ with $\omega^2 \gg 2$. Let $\ell$ be the loop whose base point is $p$. Then there is a compact contractible subset $C$ whose interior $C^{in}$ contains $\ell$. Then the following set is finite:
$$\{p(\delta) \in C^{in} | \delta \in \Delta^+(X)\}.$$

Since the fundamental group of the complement of $n$-points in $C$ is isomorphic to rank $n$ free group, we see the homotopy equivalence class of $\ell$ is uniquely given by
$$\ell_{\delta_1}^{k^1} \ell_{\delta_2}^{k_2} \cdots \ell_{\delta_m}^{k_m}$$
where each $k_i \in \mathbb{Z}$. In fact if another loop $m$ is homotopy equivalent to $\ell$ by $H : [0,1] \times [0,1] \to \mathfrak{H}_0(X)$, then there is a contractible compact set $C'$ such that $(C')^{in}$ contains the image of $H$. Since there are at most finite spherical point in $(C')^{in}$, we see the above representation is unique. Thus we have finished the proof. $\square$

To simplify the notations we denote $\ell_{v(A)}$ by $\ell_A$. By Remark 2.10, we see

$$\pi_1(\mathfrak{H}_0(X)) = \langle \ell_A | A \text{ is spherical and locally free} \rangle = \ast_A \mathbb{Z}\ell_A.$$

## 3. Hyperbolic structure on $\mathrm{Stab}^{\mathrm{n}}(X)$

Let $\mathrm{Stab}^{\dagger}(X)$ be the connected components of $\mathrm{Stab}(X)$ introduced in §2. In this section we discuss a hyperbolic structure on the normalized stability manifold $\mathrm{Stab}^{\mathrm{n}}(X)$.

To simplify explanations of this section we always use the following notations. Let $(X_i, L_i)$ $(i = 1, 2)$ be projective K3 surfaces with $\mathrm{NS}(X_i) = \mathbb{Z}L_i$ and let $\Phi : D(X_2) \to D(X_1)$ be an equivalence between them. The induced Hodge isometry $H^*(X_2, \mathbb{Z}) \to H^*(X_1, \mathbb{Z})$ by $\Phi$ is denoted by $\Phi^H$.

For a closed point $p_i \in X_i$ we set

$$v(\Phi(\mathcal{O}_{p_2})) = r_1 \oplus n_1 L_1 \oplus s_1 \text{ and } v(\Phi^{-1}(\mathcal{O}_{p_1})) = r_2 \oplus n_2 L_2 \oplus s_2.$$

Since $X_1$ and $X_2$ are Fourier-Mukai partners, we see $L_1^2 = L_2^2 = 2d$ for some $d \in \mathbb{N}$.

**Lemma 3.1.** *Notations being as above,*

(1) $r_1 = 0$ *if and only if* $r_2 = 0$. *In particular if* $r_2 = 0$ *then* $\Phi^H(\mathcal{O}_{p_2}) = \pm v(\mathcal{O}_{p_1}) = \pm(0 \oplus 0 \oplus 1)$.
(2) *If* $\Phi^H(\mathcal{O}_{p_2}) = 0 \oplus 0 \oplus 1$ *then* $\Phi^H = \big((M\otimes) \circ f_*\big)^H$ *where* $M$ *is in* $\mathrm{Pic}(X_1)$ *and* $f$ *is an isomorphism* $X_2 \to X_1$.

*Proof.* By the symmetry it is enough to show that $r_2 = 0$ under the assumption $r_1 = 0$. If $r_1 = 0$, since $v(\Phi(\mathcal{O}_{p_2}))$ is isotropic, we see $n_1^2 L_1^2 = 0$. Thus $n_1 = 0$. Moreover since $v(\Phi(\mathcal{O}_{p_2}))$ is primitive, $s_1$ should be $\pm 1$. Hence $\Phi^H(0 \oplus 0 \oplus 1) = \pm(0 \oplus 0 \oplus 1)$. Thus we have finished the proof of the first assertion.

Second assertion essentially follows from the argument in the proof for [Huy07, Corollary 10.12]. Hence we recall his arguments.

We remark that $X_2 \cong X_1$ by the global Torelli theorem and the assumption $\Phi^H(0 \oplus 0 \oplus 1) = 0 \oplus 0 \oplus 1$. One can check easily

$$v(\Phi^H(1 \oplus 0 \oplus 0)) = 1 \oplus M \oplus \frac{M^2}{2} \ (\exists M \in \mathrm{Pic}(X)),$$

by using the facts $\langle 1 \oplus 0 \oplus 0, v(\mathcal{O}_{p_2}) \rangle = -1$ and $\langle 1 \oplus 0 \oplus 0 \rangle^2 = 0$. Now consider the functor

$$\Psi = (\otimes M^{-1} \circ \Phi) : D(X_2) \to D(X_1) \to D(X_1).$$

Then we see $\Psi^H(0 \oplus 0 \oplus 1) = 0 \oplus 0 \oplus 1$ and $\Psi^H(1 \oplus 0 \oplus 0) = 1 \oplus 0 \oplus 0$. Thus $\Psi^H$ induces the isomorphism

$$\Psi^H : \mathrm{NS}(X_2) \to \mathrm{NS}(X_1).$$

Since $\mathrm{NS}(X_i) = \mathbb{Z}L_i$ we see $\Psi^H(L_2) = \pm L_1$. Since any equivalence preserves the orientations by [HMS09] we see $\Psi^H(L_2) = L_1$. This gives the proof of the second assertion. $\square$

**Lemma 3.2.** *For* $(\beta_i, \omega_i) \in \mathfrak{H}(X_i)$ $(i = 1, 2)$, *we put* $\beta_i + \sqrt{-1}\omega_i = (x_i + \sqrt{-1}y_i)L_i$.

(1) *For any* $\beta_2 + \sqrt{-1}\omega_2 \in \mathfrak{H}(X_2)$, *there exist* $\beta_1 + \sqrt{-1}\omega_1 \in \mathfrak{H}(X_1)$ *and* $\lambda \in \mathbb{C}^*$ *such that* $\Phi^H(\exp(\beta_2 + \sqrt{-1}\omega_2)) = \lambda \exp(\beta_1 + \sqrt{-1}\omega_1)$.
(2) *If* $r_1 \neq 0$ *then* $r_1 r_2 > 0$. *Furthermore we have*

$$x_1 + \sqrt{-1}y_1 = \frac{1}{d\sqrt{r_1 r_2}} \cdot \frac{-1}{(x_2 + \sqrt{-1}y_2) - \frac{n_2}{r_2}} + \frac{n_1}{r_1}.$$

*In particular this gives a linear fractional transformation on* $\mathbb{H}$.

*Proof.* We put $\mho_2 = \exp(\beta_2 + \sqrt{-1}\omega_2)$ and $\Phi^H(\mho_2) = u \oplus v \oplus w$. Since we have $\mho_2^2 = 0$ and $\mho_2 \bar{\mho}_2 > 0$, we see the following:

(a) $v^2 = 2uw$ and
(b) $v\bar{v} - u\bar{w} - \bar{u}w > 0$.

If $u = 0$ then $v^2$ should be 0. Since we have $v^2 \geq 0$ by the assumption, we see $\Phi^H(\mho_2) = 0 \oplus 0 \oplus w$. This contradicts the second inequality. Thus $u$ should not be 0 and we see

$$\begin{aligned} \Phi^H(\mho_2) &= u(1 \oplus \frac{v}{u} \oplus \frac{w}{u}) \\ &= u\Big(1 \oplus \frac{v}{u} \oplus \frac{1}{2}\Big(\frac{v}{u}\Big)^2\Big). \end{aligned}$$

Since $\frac{v}{u}$ is in $\mathrm{NS}(X) \otimes \mathbb{C}$ we can put $\frac{v}{u} = (x + \sqrt{-1}y)L_1$ for some $(x, y) \in \mathbb{R}^2$. By the inequality of (b), we see $y \neq 0$. Since $\Phi$ preserves the orientation by [HMS09], we see $y > 0$. Thus we have proved the first assertion.

We prove the second assertion. By the first assertion we put

$$\Phi^H(\exp(\beta_2 + \sqrt{-1}\omega_2)) = \lambda \exp(\beta_1 + \sqrt{-1}\omega_1).$$

Then we see

$$\begin{aligned} \lambda &= -\langle \Phi^H(\exp(\beta_2 + \sqrt{-1}\omega_2)), v(\mathcal{O}_{p_1}) \rangle \\ &= -\langle \exp(\beta_2 + \sqrt{-1}\omega_2), v(\Phi^{-1}(\mathcal{O}_{p_1})) \rangle \\ &= -Z_{(\beta_2,\omega_2)}(\Phi^{-1}(\mathcal{O}_{p_1})), \end{aligned}$$

and

$$\begin{aligned} -1 &= \langle \exp(\beta_2 + \sqrt{-1}\omega_2), v(\mathcal{O}_{p_2}) \rangle \\ &= \langle \Phi^H(\exp(\beta_2 + \sqrt{-1}\omega_2)), v(\Phi(\mathcal{O}_{p_2})) \rangle \\ &= \lambda \cdot Z_{(\beta_1,\omega_1)}(\Phi(\mathcal{O}_{p_2})). \end{aligned}$$

Thus we have

$$1 = Z_{(\beta_2,\omega_2)}(\Phi^{-1}(\mathcal{O}_{p_1})) \cdot Z_{(\beta_1,\omega_1)}(\Phi(\mathcal{O}_{p_2}))$$

By Lemma 3.1 we see $r_1 \neq 0$ and $r_2 \neq 0$. Now recall Remark 2.2. Since $v(\Phi(\mathcal{O}_{p_2}))^2 = v(\Phi^{-1}(\mathcal{O}_{p_1}))^2 = 0$, we have

$$Z_{(\beta_2,\omega_2)}(\Phi^{-1}(\mathcal{O}_{p_1})) = \frac{r_2}{2}\Big(y_2 + \sqrt{-1}\big(\frac{n_2}{r_2} - x_2\big)\Big)^2 L_2^2$$

and

$$Z_{(\beta_1,\omega_1)}(\Phi(\mathcal{O}_{p_2})) = \frac{r_1}{2}\Big(y_1 + \sqrt{-1}(\frac{n_1}{r_1} - x_1)\Big)^2 L_1^2.$$

Since $L_1^2 = L_2^2 = 2d$ we see

$$(x_1 - \frac{n_1}{r_1}) + \sqrt{-1}y_1 = \frac{\pm 1}{d\sqrt{r_1 r_2}} \cdot \frac{1}{(x_2 - \frac{n_2}{r_2}) + \sqrt{-1}y_2}.$$

Since the left hand side is in the upper half plain $\mathbb{H}$, $\sqrt{r_1 r_2}$ should be a real number. Thus we see $r_1 r_2 > 0$. Furthermore, since the imaginary part of the left hand side is positive we have

$$(x_1 - \frac{n_1}{r_1}) + \sqrt{-1}y_1 = \frac{-1}{d\sqrt{r_1 r_2}} \cdot \frac{1}{(x_2 - \frac{n_2}{r_2}) + \sqrt{-1}y_2}.$$

Thus we have finished the proof. □

Recall that $\mathrm{Stab}^{\mathrm{n}}(X) = \mathrm{Stab}^{\dagger}(X)/\tilde{GL}^{+}(2,\mathbb{R})$.

**Theorem 3.3.** *Assume that $\rho(X) = 1$.*

(1) $\mathrm{Stab}^{\mathrm{n}}(X)$ *is a hyperbolic 2 dimensional manifold.*
(2) *Let $Y$ be a Fourier-Mukai partner of $X$ and $\Phi : D(Y) \to D(X)$ an equivalence. Suppose that $\Phi$ preserves the distinguished component. Then the induced morphism $\Phi_*^{\mathrm{n}} : \mathrm{Stab}^{\mathrm{n}}(Y) \to \mathrm{Stab}^{\mathrm{n}}(X)$ is an isometry with respect to the hyperbolic metric.*

*Proof.* By Corollary 2.6, we have the covering map

$$\pi^{\mathrm{n}} : \mathrm{Stab}^{\mathrm{n}}(X) \to \mathfrak{D}_0^+(X).$$

Since $\mathfrak{D}_0^+(X)$ is isomorphic to the open subset of $\mathbb{H}$ by Lemma 2.12, we can define the hyperbolic metric on $\mathfrak{D}_0^+(X)$ which is given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

where $x + \sqrt{-1}y \in \mathbb{H}$. Since $\pi^{\mathrm{n}}$ is a covering map, we can also define the hyperbolic metric on $\mathrm{Stab}^{\mathrm{n}}(X)$. Thus $\mathrm{Stab}^{\mathrm{n}}(X)$ is hyperbolic.

Now we prove the second assertion. If $v(\Phi(\mathcal{O}_y))$ is not $\pm(0 \oplus 0 \oplus 1)$ by Lemma 3.2, we see that the induced morphism between $\mathfrak{D}_0^+(Y) \to \mathfrak{D}_0^+(X)$ is given by the linearly fractional transformation. Since $\pi^{\mathrm{n}}$ is an isometry, $\Phi_*^{\mathrm{n}}$ is also an isometry. Suppose that $v(\Phi(\mathcal{O}_y)) = \pm(0 \oplus 0 \oplus 1)$. If necessary by taking a shift [1] which gives the trivial action on $\mathfrak{H}(X)$ we can assume that $v(\Phi(\mathcal{O}_y)) = 0 \oplus 0 \oplus 1$. Then, by Lemma 3.1, the induced action on $\mathbb{H}$ is given by a parallel transformation $z \mapsto z + n$ for some $n \in \mathbb{Z}$. Thus we have finished the proof. □

## 4. Simply connectedness of $\mathrm{Stab}^{\mathrm{n}}(X)$

In this section we always assume $\rho(X) = 1$. Then, as was shown in the previous section, $\mathrm{Stab}^{\mathrm{n}}(X)$ is a hyperbolic manifold. By using the hyperbolic structure, we shall discuss the simply connectedness of $\mathrm{Stab}^{\dagger}(X)$. Namely we show the following:

**Theorem 4.1.** *The following conditions are equivalent.*

1. $\mathrm{Stab}^{\dagger}(X)$ *is simply connected.*
2. $\mathrm{Stab}^{\mathrm{n}}(X)$ *is isomorphic to the upper half plain* $\mathbb{H}$.
3. $W(X)$ *is isomorphic to the free group generated by* $T_A^2$*:*
$$W(X) = \ast_A(\mathbb{Z} \cdot T_A^2),$$
*where* $A$ *runs through all spherical locally free sheaves.*

*Proof.* We first show that $\mathrm{Stab}^{\dagger}(X)$ is simply connected if and only if $\mathrm{Stab}^{\mathrm{n}}(X)$ is simply connected. Since the right action of $\tilde{GL}^+(2,\mathbb{R})$ on $\mathrm{Stab}^{\dagger}(X)$ is free, the natural map
$$\mathrm{Stab}^{\dagger}(X) \to \mathrm{Stab}^{\mathrm{n}}(X)$$
gives the $\tilde{GL}^+(2,\mathbb{R})$-bundle on $\mathrm{Stab}^{\mathrm{n}}(X)$. Thus there is an exact sequence of fundamental groups:
$$\pi_1(\tilde{GL}^+(2,\mathbb{R})) \longrightarrow \pi_1(\mathrm{Stab}^{\dagger}(X)) \longrightarrow \pi_1(\mathrm{Stab}^{\mathrm{n}}(X)) \longrightarrow 1.$$
Since $\tilde{GL}^+(2,\mathbb{R})$ is simply connected we see that $\pi_1(\mathrm{Stab}^{\dagger}(X)) = \{1\}$ if and only if $\pi_1(\mathrm{Stab}^{\mathrm{n}}(X)) = \{1\}$.

Since $\mathrm{Stab}^{\mathrm{n}}(X)$ is a hyperbolic and complex manifold, $\mathrm{Stab}^{\mathrm{n}}(X)$ is isomorphic to $\mathbb{H}$ if and only if $\pi_1(\mathrm{Stab}^{\mathrm{n}}(X)) = \{1\}$ by Riemann's mapping theorem. Thus we have proved that the first condition is equivalent to the second one.

We secondly show the first condition is equivalent to the third one. Let $\mathrm{Cov}(\pi)$ be the covering transformation group of $\pi : \mathrm{Stab}^{\dagger}(X) \to \mathcal{P}_0^+(X)$. We put $\tilde{W}(X)$ by the group generated by $W(X)$ and the double shift [2]. Note that $\tilde{W}(X)$ is isomorphic to $W(X) \times \mathbb{Z} \cdot [2]$.

We claim that $\tilde{W}(X)$ is isomorphic to $\mathrm{Cov}(\pi)$. Recall that all spherical sheaf $A$ on $X$ with $\rho(X) = 1$ is $\mu$-stable by Remark 2.10. Hence any $\Phi \in \tilde{W}(X)$ gives a trivial action on $H^*(X,\mathbb{Z})$ and preserves the connected component $\mathrm{Stab}^{\dagger}(X)$. Thus $\Phi$ gives the covering transformation by [Bri08, Theorem 13.3]. Thus we have the group homomorphism $\tilde{W}(X) \to \mathrm{Cov}(X)$. In particular by Proposition 2.4, we see this morphism is a surjection. Furthermore as is shown in [Bri08, Theorem 13.3], this is injective. Thus we have proved our claim.

Since the covering $\pi : \mathrm{Stab}^{\dagger}(X) \to \mathcal{P}_0^+(X)$ is a Galois covering, we have the exact sequence of groups:
$$1 \longrightarrow \pi_1(\mathrm{Stab}^{\dagger}(X)) \longrightarrow \pi_1(\mathcal{P}_0^+(X)) \xrightarrow{\varphi} \mathrm{Cov}(\pi) \longrightarrow 1.$$

As will be shown in the Proposition 5.4 we see $\varphi(\ell_A) = T_A^2$ and $\varphi(g) = [2]$. If $\mathrm{Stab}^\dagger(X)$ is simply connected then $\varphi$ is the isomorphism. Hence $W(X)$ is a free group generated by $T_A^2$. Conversely if $W(X)$ is a free group generated by $T_A^2$, then $\varphi$ is an isomorphism. Hence $\mathrm{Stab}^\dagger(X)$ is simply connected. □

**Remark 4.2.** Since the quotient map $\mathrm{Stab}^\dagger(X) \to \mathrm{Stab}^{\mathrm{n}}(X)$ is a $\tilde{GL}^+(2,\mathbb{R})$-bundle, we see that $\mathrm{Stab}^\dagger(X)$ is simply connected if and only if $\mathrm{Stab}^\dagger(X)$ is a $\tilde{GL}^+(2,\mathbb{R})$-bundle over $\mathbb{H}$. Thus we can deduce the global geometry of the stability manifold $\mathrm{Stab}^\dagger(X)$.

**Remark 4.3.** We give some remarks for $W(X)$. Recall that any equivalence $\Phi \in \mathrm{Aut}(D(X))$ induces the Hodge isometry $\Phi^H$ of $H^*(X,\mathbb{Z})$ in a canonical way. If Bridgeland's conjecture holds, the group $W(X) \times \mathbb{Z}[2]$ is the kernel $\mathrm{Ker}(\rho)$ of the natural map

$$\rho : \mathrm{Aut}(D(X)) \to O^+_{\mathrm{Hodge}}(H^*(X,\mathbb{Z})) : \Phi \to \Phi^H.$$

Moreover $\mathrm{Ker}(\rho)$ is given by $\pi_1(\mathcal{P}_0^+(X))$. The freeness of $W(X)$ means any two orthogonal complements $\langle\delta_1\rangle^\perp$ and $\langle\delta_2\rangle^\perp$ (where $\delta_1$ and $\delta_2 \in \Delta(X)$) do not meet each other in $\mathcal{P}_0^+(X)$.

In more general situation (namely the case of $\rho(X) \geq 2$) there should be some orthogonal complements such that $\langle\delta_1\rangle^\perp$ and $\langle\delta_2\rangle^\perp$ meet each other. Hence we expect that the quotient group $\mathrm{Ker}(\rho)/\mathbb{Z}\cdot[2]$ is not a free group.

## 5. Wall and the hyperbolic structure

Let $X$ be a projective K3 surface with Picard rank one. We have two goals of this section. The first aim is to show Proposition 5.4 which is necessary for Theorem 4.1. The second aim is to show that any wall is geodesic.

Now we start this section from the following key lemma.

**Lemma 5.1.** *Any $\sigma \in \partial U(X)$ is in a general potion (See also Definition B.4).*

Before we start the proof, we remark that Maciocia proved a similar assertion in a slightly different situation in [Mac12].

*Proof.* Suppose that there is an element $\sigma = (\mathcal{A}, Z) \in \partial U(X)$ which is not general. Let $W_1$ and $W_2$ be two irreducible components of $\partial U(X)$ such that $\sigma \in W_1 \cap W_2$. By Proposition B.3 we may assume $\forall\tau_1 \in W_1 \setminus \{\sigma\}$ and $\forall\tau_2 \in W_2 \setminus \{\sigma\}$ are in general positions in a sufficiently small neighborhood of $\sigma$. Hence by Theorem B.5 there are two $(-2)$-vectors $\delta_i \in \Delta^+(X)$ $(i = 1,2)$ such that for any $\tau_i = (\mathcal{A}_i, Z_i) \in W_i \setminus \{\sigma\}$ the imaginary part $\mathfrak{Im} Z_i(\mathcal{O}_x)\overline{Z_i(\delta_i)}$ is 0 where $i \in \{1,2\}$ and $x \in X$. Since these are closed conditions, the central charge $Z$ of $\sigma$ also satisfies the following condition:

$$\mathfrak{Im} Z(\mathcal{O}_x)\overline{Z(\delta_1)} = \mathfrak{Im} Z(\mathcal{O}_x)\overline{Z(\delta_2)} = 0. \tag{5.1}$$

By the assumption $\mathrm{NS}(X) = \mathbb{Z}L$, there exists $g \in GL^+(2,\mathbb{R})$ such that $Z'(E) := g^{-1} \circ Z(E) = \langle \exp(\beta + \sqrt{-1}\omega), v(E)\rangle$ where $\beta \in \mathrm{NS}(X)_{\mathbb{R}}$ and $\omega \in \mathrm{Amp}(X)$.

Now we put $\delta_i = r_i \oplus n_i L \oplus s_i$. Note that $r_i \neq 0$ since $n_i^2 L_i^2 \geq 0$. Since $Z'(\mathcal{O}_x) = -1$ we see $\mathfrak{Im} Z'(\delta_i)$ is zero by the condition (5.1). Thus we see

$$\frac{n_1 L}{r_1} = \frac{n_2 L}{r_2} = \beta.$$

Since $\delta_i^2 = -2$ we see $\gcd(r_i, n_i) = 1$. Hence we have $\delta_1 = \delta_2$. This contradicts $W_1 \neq W_2$. $\square$

By Lemma 5.1 and Theorem B.5 we see $\partial U(X)$ is a disjoint union of real codimension 1 submanifolds:

$$\partial U(X) = \coprod_{A:\text{spherical locally free}} (W_A^+ \sqcup W_A^-),$$

where $W_A^+$ (respectively $W_A^-$) is the set of stability conditions whose type is $(A^+)$ (respectively $(A^-)$). In the following we give an explicit description of each component $W_A^{\pm}$.

**Lemma 5.2.** *Let $X$ be a projective K3 surface with $\mathrm{NS}(X) = \mathbb{Z}L$ and let $A$ be a spherical locally free sheaf. We put $v(A) = r_A \oplus n_A L \oplus s_A$ and define the set $S(v(A))$ by*

$$S(v(A)) = \{(\beta, \omega) \in \mathfrak{H}(X) | \beta = \frac{n_A L}{r_A}, 0 < \omega^2 < \frac{2}{r_A^2}\}.$$

*Then $W_A^{\pm}$ is isomorphic to $S(v(A)) \times \tilde{GL}^+(2,\mathbb{R})$. In particular $W_A^{\pm}/\tilde{GL}^+(2,\mathbb{R})$ is a hyperbolic line in $\mathrm{Stab}^{\mathrm{n}}(X)$ which is isomorphic to $S(v(A))$.*

*Proof.* We have to consider two cases: $\sigma \in W_A^+$ or $\sigma \in W_A^-$. Since the proof is similar, we give the proof only for the case $\sigma \in W_A^+$.

Since $\sigma \in W_A^+$, the Jordan-Hölder filtration of $\mathcal{O}_x$ is given by the spherical triangle (2.1)

$$A^{\oplus r_A} \longrightarrow \mathcal{O}_x \longrightarrow T_A(\mathcal{O}_x). \tag{5.2}$$

By taking $T_A^{-1}$ to the triangle (5.2) we have

$$A^{\oplus r_A}[1] \longrightarrow T_A^{-1}(\mathcal{O}_x) \longrightarrow \mathcal{O}_x. \tag{5.3}$$

Thus $\mathcal{O}_x$ is $T_{A*}^{-1}\sigma$-stable. Hence $T_{A*}^{-1}\sigma$ is in $U(X)$.

Now we put $T_{A*}^{-1}\sigma = \tau = (\mathcal{A}, Z)$. Since $Z(A[1])/Z(\mathcal{O}_x) \in \mathbb{R}_{>0}$, we see that $\tau$ is in the set

$$W' = \{\sigma_{(\beta,\omega)} \in V(X) | \beta = \frac{n_A L}{r_A}, \frac{2}{r_A^2} < \omega^2\} \cdot \tilde{GL}^+(2,\mathbb{R}).$$

Thus we see $W_A^+ \subset T_{A*}W'$. To show the inverse inclusion, let $\tau' = (\mathcal{A}', Z')$ be in $W'$. As we remarked in Remark 2.10, $A$ is $\mu$-stable locally free sheaf. Then $A[1]$ has no nontrivial subobject in $\mathcal{A}'$ by [Huy08, Theorem 0.2]. Hence $A[1]$ is $\tau'$-stable, in particular, with phase 1. Since $T_A^{-1}(\mathcal{O}_x)$ is given by

the extension (5.3) of $\mathcal{O}_x$ and $A^{\oplus r_A}[1]$, the object $T_A^{-1}(\mathcal{O}_x)$ is strictly $\tau'$-semistable. Thus by taking $T_A$ to the triangle (5.3), we obtain the Jordan-Hölder filtration (5.2). Hence we see $W_A^+ = T_{A*}W'$.

Since the induced morphism between $\mathfrak{H}(X)$ by $T_A$ is given by Lemma 3.2, we see

$$W_A^+ = T_{A*}W' \cong S(v(A)) \times \tilde{GL}^+(2, \mathbb{R}).$$

□

For a spherical locally free sheaf $A$ we define the point $q = p(T_A(\mathcal{O}_x)) \in \bar{\mathfrak{H}}(X)$ by $(\beta, \omega) = (\frac{c_1(A)}{r_A}, 0)$. By the simple calculation we see that

$$\langle \exp(q), v(T_A(\mathcal{O}_x)) \rangle = 0.$$

Thus in the sense of Definition 2.7, $p(T_A(\mathcal{O}_x))$ could be regarded as the associated point of the isotropic vector $v(T_A(\mathcal{O}_x))$. In view of this we define the following notion:

**Definition 5.3.** An associated point $p \in \bar{\mathfrak{H}}(X)$ with a primitive isotropic vector $v \in \mathcal{N}(X)$ is the point which satisfies

$$\langle \exp(p), v \rangle = 0.$$

Clearly if $v = r \oplus nL \oplus s$ then $p$ is given by $\frac{n}{r}$. In particuclar if $v = 0 \oplus 0 \oplus 1$ the associated point is $\infty \in \bar{\mathfrak{H}}(X)$. We denote the point by $p(v)$.

As an application of Lemma 5.2 we give the proof of a remained proposition:

**Proposition 5.4.** *Let* $\varphi : \pi_1(\mathcal{P}_0^+(X)) \to \mathrm{Cov}(\pi)$ *be the morphism in the proof of Theorem 4.1. Then* $\varphi(\ell_A) = T_A^2$ *and* $\varphi(g) = [2]$.

*Proof.* We set a base point of $\pi_1(\mathfrak{H}_0(X))$ as $\sqrt{-1}\omega_0$ with $\omega_0^2 \gg 2$. We also define a base point of $\pi_1(\mathcal{P}_0^+(X))$ by $\exp(\sqrt{-1}\omega_0)$. Let $\sigma_0 = \sigma_{(0,\omega_0)} \in V(X)$ be a base point of the covering map $\pi : \mathrm{Stab}^\dagger(X) \to \mathcal{P}_0^+(X)$.

Let $\ell_A : [0,1] \to \mathfrak{H}_0(X)$ be the loop defined in Definition 2.13 which turns round the point $p(v(A))$ and let $\tilde{\ell}_A$ be the lift of $\ell_A$ to $\mathrm{Stab}^\dagger(X)$.

The second assertion is almost obvious. In Definitions 2.13 we choosed $g$ as

$$g : [0,1] \to GL^+(2, \mathbb{R}) : t \mapsto \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}.$$

Then the induced action of $g$ on $\mathrm{Stab}^\dagger(X)$ is given by the double shift [2]. Hence it is enough to show that $\tilde{\ell}_A(1) = T_{A*}^2\sigma_0$.

Since there are no spherical point $p(\delta)$ inside the loop $\ell_A$ except for $p(v(A))$ itself, the intersection $\ell_A([0,1]) \cap \pi(\partial U(X))$ consists of only one point. We may assume the point is given by $\ell_A(1/2)$. Since we have $\tilde{\ell}_A([0,1/2)) \subset U(X)$ we see that $\tilde{\ell}_A(1/2) = \tau$ is in $\partial U(X)$ and that $\tau$ is of type $(A^+)$ or $(A^-)$ by Lemma 5.1 and Theorem B.5.

We finally claim that $\tau$ is of type $(A^+)$. To prove the claim we put

$$\tilde{\ell}_A\Big(\frac{1}{2}-\epsilon\Big)=\sigma_\epsilon=(\mathcal{A}_\epsilon,Z_\epsilon)\in \mathrm{Stab}^\dagger(X),$$

for $0<\epsilon\ll 1$. In fact suppose to the contrary that $\tau$ is of type $(A^-)$. By Proposition B.3 (2) we may assume both $A$ and $T_A^{-1}(\mathcal{O}_x)$ are $\sigma_\epsilon$-stable for any $\epsilon$. Since we see $\mathfrak{Im} Z_\epsilon(\mathcal{O}_x)/Z_\epsilon(A[2])>0$, the distinguished triangle

$$T_A^{-1}(\mathcal{O}_x)\longrightarrow \mathcal{O}_x\longrightarrow A^{\oplus r_A}[2]$$

gives the Harder-Narashimhan filtration of $\mathcal{O}_x$ in $\sigma_\epsilon$. This contradicts the fact that $\mathcal{O}_x$ is $\sigma_\epsilon$-stable. Hence $\ell_A(1/2)$ is of type $(A^+)$ and $\tilde{\ell}_A(1/2+\epsilon)$ is in $T_{A*}^2U(X)$. For $t>1/2$, since $\ell_A$ does not meet $\pi(\partial U(X))$, we see $\tilde{\ell}_A(1)=T^2_{A*}\sigma_0$. □

Finally we observe so called walls in terms of the hyperbolic structure. As we showed in Lemma 5.2 each boundary components of $\partial V(X)$ is geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$. More generally we show that any wall is geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$.

Let $\mathcal{S}$ be the set objects which have bounded mass in $\mathrm{Stab}^\dagger(X)$, and $B$ a compact subset of $\mathrm{Stab}^\dagger(X)$ (See also Appendix B). Then by Proposition B.3 we have a finite set $\{W_\gamma\}_{\gamma\in\Gamma}$ of real codimension 1 submanifolds satisfying the property in the proposition. For the set $\{W_\gamma\}_{\gamma\in\Gamma}$ we put

$$\mathfrak{W}(\mathcal{S},B)=\Big(\bigcup_{\gamma\in\Gamma}W_\gamma\Big)/\tilde{GL}^+(2,\mathbb{R}).$$

Note that $\mathfrak{W}(\mathcal{S},B)$ is a subset of $\mathrm{Stab}^{\mathrm{n}}(X)$.

**Theorem 5.5.** *The set* $\mathfrak{W}(\mathcal{S},B)$ *is geodesic in* $\mathrm{Stab}^{\mathrm{n}}(X)$.

*Proof.* Following Proposition B.3 let $\mathcal{T}$ be the set of objects

$$\mathcal{T}=\{A\in D(X)|\exists E\in\mathcal{S},\exists\sigma\in B \text{ such that } m_\sigma(A)\le m_\sigma(E)\}.$$

We put the set of Mukai vectors in $\mathcal{T}$ by $I=\{v(A)|A\in\mathcal{T}\}$ and let $\Gamma$ be the pair of $(v_i,v_j)\in I\times I$ which are not proportional. As was shown in Proposition B.3, each wall component $W_\gamma$ is given by

$$W_\gamma=\{\sigma=(\mathcal{A},Z)\in\mathrm{Stab}^\dagger(X)|Z(v_i)/Z(v_j)\in\mathbb{R}_{>0}\}.$$

We put $W_\gamma/\tilde{GL}^+(2,\mathbb{R})$ by $\mathfrak{W}_\gamma$. It is enough to prove that $\mathfrak{W}_\gamma$ is geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$.

Since $I$ is finite set (Recall that $\mathcal{T}$ has bounded mass) we can take a sufficiently large $m\in\mathbb{Z}$ so that the rank of all vectors in $T^H_{mL}(I)$ are not 0. For the set $T^H_{mL}(I)$ we define $\mathfrak{W}^T_\gamma$ by

$$\mathfrak{W}^T_\gamma=\{[\sigma]=[(\mathcal{A},Z)]\in\mathrm{Stab}^{\mathrm{n}}(X)|Z(T^H_{mL}(v_i))/Z(T^H_{mL}(v_j))\in\mathbb{R}_{>0}\}.$$

We may assume the central charge of $[\sigma]\in\mathfrak{W}^T_\gamma$ is given by

$$Z(E)=\langle\exp(\beta+\sqrt{-1}\omega),v(E)\rangle$$

where $(\beta,\omega)\in\mathfrak{H}(X)$.

We note that $\sigma \in \mathfrak{W}_\gamma^T$ satisfies the following equation

$$\mathfrak{Im} Z(T_{mL}^H(v_i))\overline{Z(T_{mL}^H(v_j))} = 0. \tag{5.4}$$

Then one can easily check that the equation (5.4) defines hyperbolic line in $\mathfrak{H}(X)$. Since the hyperbolic structure is induced from $\mathfrak{H}(X)$ the set $\mathfrak{W}_\gamma^T$ is geodesic also in $\mathrm{Stab}^{\mathrm{n}}(X)$. Since we have $T_{mL}^{\mathrm{n}}\mathfrak{W}_\gamma^T = \mathfrak{W}_\gamma$ the set $\mathfrak{W}_\gamma$ is also geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$ by Theorem 3.3. □

## 6. Revisit of Orlov's theorem via hyperbolic structure

In this section we demonstrate an application of the hyperbolic structure on $\mathrm{Stab}^{\mathrm{n}}(X)$. Namely we prove Orlov's theorem without the global Torelli theorem but with assuming the connectedness of $\mathrm{Stab}(X)$. Hence our application suggests that Bridgeland's theory substitutes for the global Torelli theorem.

After we prepare some lemmas, we give another proof of Orlov's theorem. Throughout this section we use the following notations.

For a K3 surface $(X, L)$ with $\rho(X) = 1$ we put $L^2 = 2d$. Suppose that $E \in D(X)$ satisfies $v(E)^2 = 0$ and $A \in D(X)$ is spherical. We put their Mukai vectors respectively

$$v(E) = r_E \oplus n_E L \oplus s_E \text{ and } v(A) = r_A \oplus n_A L \oplus s_A.$$

We denote $(\beta, \omega) \in \bar{\mathfrak{H}}(X)$ by $(xL, yL)$.

The main object of the first half is the following set

$$\mathfrak{W}(A, E) = \{(\beta, \omega) \in \bar{\mathfrak{H}}(X) | \mathfrak{Im} Z_{(\beta,\omega)}(E)\overline{Z_{(\beta,\omega)}(A)} = 0\}.$$

One can easily check that the condition $\mathfrak{Im} Z_{(\beta,\omega)}(E)\overline{Z_{(\beta,\omega)}(A)} = 0$ is equivalent to

$$N_{A,E}(x, y) = \lambda_E(\frac{-1}{r_A} + dr_A y^2 - \frac{d\lambda_A^2}{r_A}) - \lambda_A(dr_E y^2 - \frac{\lambda_E^2}{r_E}) = 0,$$

where $\lambda_E = n_E - r_E x$ and $\lambda_A = n_A - r_A x$. We also have

$$N_{A,E}(x, y) = d(r_A n_E - r_E n_A)y^2 + d\lambda_E \lambda_A(\frac{n_E}{r_E} - \frac{n_A}{r_A}) - \frac{\lambda_E}{r_A}. \tag{6.1}$$

**Lemma 6.1.** *Suppose that $r_E > 0$ and $\frac{n_E}{r_E} \neq \frac{n_A}{r_A}$. Then $\mathfrak{W}(A, E)$ is the half circle passing through the following four points:*

$$(\beta, \omega) = (\alpha_E, 0), (\frac{n_E}{r_E}, 0), (\frac{n_A}{r_A}, \frac{1}{\sqrt{d}|r_A|}) \textit{ and } (\alpha_A, \frac{1}{\sqrt{d}|r_A|}),$$

*where $\alpha_E = \frac{n_A}{r_A} - \frac{1}{dr_A^2(\frac{n_E}{r_E} - \frac{n_A}{r_A})}$ and $\alpha_A = \frac{n_E}{r_E} - \frac{1}{dr_A^2(\frac{n_E}{r_E} - \frac{n_A}{r_A})}$.*

We can prove Lemma 6.1 by the simple calculation of (6.1). Hence we omit the proof. In particular the first two points are associated points with respectively $T_A(E)$ and $E$. Hence we put them respectively

- $p(T_A(E)) = (\alpha_E, 0)$,
- $p(E) = (\frac{n_E}{r_E}, 0)$,

- $p(A) = (\frac{n_A}{r_A}, \frac{1}{\sqrt{d}|r_A|})$ and
- $q = (\alpha_A, \frac{1}{\sqrt{d}|r_A|})$.

We remark that if $\frac{n_E}{r_E} = \frac{n_A}{r_A}$ then $\mathfrak{W}(A,E)$ is a hyperbolic line defined by $x = \frac{n_E}{r_E}$.

**Lemma 6.2.** *Suppose that $r_E > 0$ and $\frac{n_E}{r_E} - \frac{n_A}{r_A} > 0$. Then there two types of the configuration of the above four points on $\mathfrak{W}(A,E)$:*

(I) *If $\frac{1}{d|r_A|} \leq \frac{n_E}{r_E} - \frac{n_A}{r_A}$ then we have $\alpha_E < \frac{n_A}{r_A} \leq \alpha_A < \frac{n_E}{r_E}$. See also Figure 2.*
(II) *If $0 < \frac{n_E}{r_E} - \frac{n_A}{r_A} < \frac{1}{d|r_A|}$ then we have $\alpha_E < \alpha_A < \frac{n_A}{r_A} < \frac{n_E}{r_E}$. See also Figure 3.*

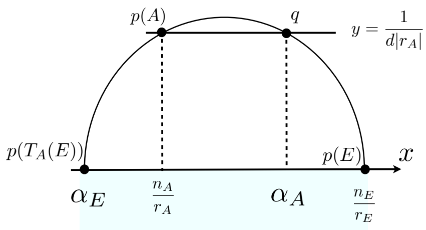


FIGURE 2. figure for type (I) in Lemma 6.2

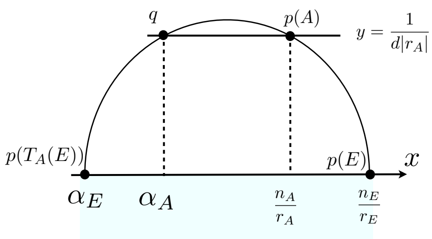


FIGURE 3. figure for for type (II) in Lemma 6.2

Similarly to Lemma 6.1 we could prove Lemma 6.2 by simple calculations. Now we give an upper bound of the diameter of the half circle $\mathfrak{W}(A,E)$. Clearly from Lemma 6.1 the diameter is given by $\frac{n_E}{r_E} - \alpha_E$.

**Proposition 6.3.** *Suppose that $r_E > 0$ and $0 < \frac{n_E}{r_E} - \frac{n_A}{r_A} < \frac{1}{\sqrt{d}|r_A|}$. Then we have*

$$0 < \frac{n_E}{r_E} - \alpha_E \leq \frac{1}{r_E} + \frac{r_E}{d}.$$

*Proof.* By the assumption one easily sees $r_A \cdot (r_A n_E - r_E n_A) > 0$. Hence we see

$$\begin{aligned} \frac{n_E}{r_E} - \alpha_E &= \Big(\frac{n_E}{r_E} - \frac{n_A}{r_A}\Big) + \frac{1}{dr_A^2\Big(\frac{n_E}{r_E} - \frac{n_A}{r_A}\Big)} \\ &= \Big|\frac{1}{r_A}\Big| \cdot \Big(\frac{|r_A n_E - r_E n_A|}{r_E} + \frac{r_E}{d|r_A n_E - r_E n_A|}\Big) \\ &\leq \frac{|r_A n_E - r_E n_A|}{r_E} + \frac{r_E}{d|r_A n_E - r_E n_A|}. \end{aligned} \tag{6.2}$$

By the assumption we have

$$\frac{|r_A n_E - r_E n_A|}{r_E} < \frac{r_E}{d|r_A n_E - r_E n_A|}.$$

Since the continuous function $f(t) = \frac{1}{t} + \frac{t}{d}$ on $\mathbb{R}_{>0}$ is an increasing function for $\frac{1}{t} < \frac{t}{d}$. Since we have $\frac{r_E}{|r_A n_E - r_E n_A|} \leq r_E$ the following inequality holds:

$$(6.2) \leq \frac{1}{r_E} + \frac{r_E}{d}.$$

Thus we have proved the inequality. □

Finally we prove the main result of this section.

**Proposition 6.4.** *Let $(X, L_X)$ be a projective K3 surface with $\rho(X) = 1$ and $(Y, L_Y)$ a Fourier-Mukai partner of $(X, L)$. If an equivalence $\Phi : D(Y) \to D(X)$ preserves the distinguished component, then $Y$ is isomorphic to the fine moduli space of Gieseker stable torsion free sheaves.*

*Proof.* We first put $L_X^2 = L_Y^2 = 2d$ and $v_0 = v(\Phi(\mathcal{O}_y)) = r \oplus nL_X \oplus s$. If necessary by taking $T_{\mathcal{O}_X}$ and [1], we may assume $r > 0$. We denote the composition of two morphisms $\mathrm{Stab}^\dagger(X) \to \mathcal{P}_0^+(X) \to \mathfrak{H}_0(X)$ by $\pi_{\mathfrak{D}}$. By the assumption we have $\Phi_* U(Y) \subset \mathrm{Stab}^\dagger(X)$.

We take a stability condition $\tau \in U(Y)$ so that $\pi_{\mathfrak{D}}(\tau) = (\beta_0, \omega_0) = (aL_X, bL_X)$ with
 (i) $\frac{1}{r} + \frac{r}{d} < \frac{n}{r} - a$ and
 (ii) $2 < \omega_0^2$.

By the second condition (ii) and Lemma 5.2 we see $\pi_{\mathfrak{D}} \circ \Phi_*(\tau)$ does not lie on $\pi_{\mathfrak{D}}(\partial U(X))$. Hence $\Phi_*(\tau)$ is in a chamber of $\mathrm{Stab}^\dagger(X)$ by Proposition 2.4. Namely we see the following:

$$\exists \Psi \in W(X) \times \mathbb{Z}[2] \text{ such that } (\Psi \circ \Phi)_*(\tau) \in U(X).$$

Now we put $\Phi' = \Psi \circ \Phi$ and take $\sigma_0 \in V(X)$ as $\sigma_{(\beta_0, \omega_0)}$. Since $\Phi'_*(\tau)$ and $\sigma_0$ belong to the same $\tilde{GL}^+(2, \mathbb{R})$-orbit, $\sigma_0$ is in $V(X) \cap \Phi'_*(U(Y))$. We define a family $\mathcal{F}$ of stability conditions as follows:

$$\mathcal{F} = \{\sigma_{(\beta_0, t\omega_0)} \in V(X) | 1 < t \in \mathbb{R}\}.$$

By Lemma 6.2 and Proposition 6.3, one can show the following claim:

**Claim :** $\pi_{\mathfrak{D}}(\mathcal{F}) \cap \pi_{\mathfrak{D}} \circ \Phi'_*(\partial U(Y)) = \emptyset$.

The proof of this claim is given in Lemma 6.5 (below).

By the claim $\mathcal{F}$ does not meet $\Phi'_*(\partial U(Y))$. Hence $\mathcal{F} \subset \Phi'_*(U(Y))$ and the object $\Phi'(\mathcal{O}_y)$ is $\sigma$-stable for all $\sigma \in \mathcal{F}$. By Bridgeland's large volume limit argument [Bri08, Proposition 14.2] we see that $\Phi'(\mathcal{O}_y)$ is a Gieseker semistable torsion free sheaf. Moreover by [Muk87, Proposition 3.14] (or the argument of [Kaw10, Lemma 4.1]) $\Phi'(\mathcal{O}_y)$ is Gieseker stable. Since $v_0 = v(\Phi'(\mathcal{O}_y))$ is isotropic and there is $u \in \mathcal{N}(X)$ such that $\langle v_0, u\rangle = 1$, there exists the fine moduli space $\mathcal{M}$ of Gieseker stable sheaves (See also [Huy07, Lemma 10.22 and Proposition 10.20]). Hence $Y$ is isomorphic to $\mathcal{M}$. □

**Lemma 6.5.** *Notations being as above,*

$$\pi_{\mathfrak{D}}(\mathcal{F}) \cap \pi_{\mathfrak{D}} \circ \Phi'_*(\partial U(Y)) = \emptyset.$$

*Proof.* We use the same notations as in the proof of the previous proposition. In addition to them, for $\delta \in \Delta(X)$ we put

$$T_\delta(v_0) = v_0 + \langle v_0, \delta\rangle\delta.$$

By Lemma 5.2 and Theorem 3.3 each component $\mathfrak{W}$ of $\pi_{\mathfrak{D}} \circ \Phi'_*(\partial U(Y))$ is a hyperbolic segment spanned by $p(\delta)$ and $p(T_\delta(v_0))$. In particular $\mathfrak{W}$ is a subset of $\mathfrak{W}(A, \Phi(\mathcal{O}_y))$ for some spherical object $A$ with $v(A) = \pm\delta$. By Lemma 6.2 there are two types of $\mathfrak{W}$.

Suppose that $\mathfrak{W}$ is of type (I). Then for any $(\beta, \omega) \in \mathfrak{W}$ we see $\omega^2 \leq 2$. Since $\pi_{\mathfrak{D}}(\mathcal{F}) \ni (\beta_0, t\omega_0)$ satisfies $t\omega_0^2 > 2$, $\pi_{\mathfrak{D}}(\mathcal{D})$ does not meet type (I) components.

Suppose that $\mathfrak{W}$ is of type (II). By Proposition 6.3, the component $\mathfrak{W}$ is contained in the half circle whose diameter is $\frac{1}{r} + \frac{r}{d}$ and whose edge point is $p(v_0)$. For $\beta_0 = aL_X$ since we assumed

$$\frac{1}{r} + \frac{r}{d} < \frac{n}{r} - a,$$

$\mathcal{F}$ does not intersect type (II) components. □

**Remark 6.6.** Finally we explain the relation between author's work and Huybrechts's question in [Huy08].

In [Huy08, Proposition 4.1], it was proven that all non-trivial Fourier-Mukai partners of projective K3 surfaces are given by the fine moduli spaces of $\mu$-stable locally free sheaves (See also [Huy08, Proposition 4.1]). We note that this proposition holds for all projective K3 surfaces. If the Picard rank is one, the proof of the proposition is based on the lattice argument. In the proof of [Huy08, Proposition 4.1] Huybrechts asks whether there is a geometric proof.

In the previous work [Kaw11, Theorem 5.4], the author gave an answer of Huybrechts's question, that is a geometric proof. However our proof is not completely independent of lattice theories, because it is based on Orlov's theorem which strongly depends on the global Torelli theorem.

As a consequence of Proposition 6.4 and [Kaw11, Theorem 5.4], we could give the another proof of [Huy08, Proposition 4.1] which is completely independent of the global Torelli theorem with assuming the connectedness of $\mathrm{Stab}(X)$. We remark that a key ingredient of both proofs is to analyze the change of the boundary $\partial U(X)$ under Fourier-Mukai transformations. These also show how the hyperbolic structure is helpful for us.

## Appendix A. Bridgeland's stability condition

This section is a survey of the general theory of Bridgeland's stability conditions on triangulated categories. Let $\mathcal{D}$ be a $\mathbb{C}$ linear triangulated category. The symbol [1] means the shift of $\mathcal{D}$ and $[n]$ means the $n$-times composition of [1].

**Definition A.1.** Let $\sigma = (Z, \mathcal{P})$ be a pair consisting of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ from the Grothendieck group of $\mathcal{D}$ to $\mathbb{C}$ which is called a *central charge*, and a collection $\mathcal{P} = \{\mathcal{P}(\phi)\}$ of additive full subcategories $\mathcal{P}(\phi)$ of $\mathcal{D}$ parametrized by the real numbers $\phi$. This pair $\sigma$ is a stability condition on $\mathcal{D}$ if it is satisfied the following condition:

(1) If $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m_E \exp(\sqrt{-1}\pi\phi)$ where $m_E > 0$.
(2) If $\phi > \psi$, then $\mathrm{Hom}_{\mathcal{D}}(E, F) = 0$ for all $E \in \mathcal{P}(\phi)$ and $F \in \mathcal{P}(\psi)$.
(3) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
(4) For all $0 \neq E \in \mathcal{D}$, there is a sequence of distinguished triangles satisfying the following condition:

(A.1)
$$\begin{tikzcd}
0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E, \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \arrow[ul, dashed, "{[1]}"] & & & & A_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

where each $A_i$ is in $\mathcal{P}(\phi_i)$ $(i = 1, \cdots n)$ with $\phi_1 > \cdots > \phi_n$.

**Remark A.2.** Here we collect some facts and definitions.

(1) Each $\mathcal{P}(\phi)$ is an abelian category.
(2) By definition, for each $0 \neq E \in \mathcal{D}$, there is at most one $\phi \in \mathbb{R}$ such that $E \in \mathcal{P}(\phi)$. When $E \in \mathcal{P}(\phi)$, we define $\arg Z(E) := \phi$ and call $\phi$ the *phase* of $E$.
(3) $E \in \mathcal{D}$ is said to be $\sigma$*-semistable* when $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$. In particular, if $E$ is minimal in $\mathcal{P}(\phi)$ (that is, $E$ has no non-trivial subobjects) then $E$ is said to be $\sigma$*-stable*.
(4) The sequence (A.1) is unique up to isomorphism. We can easily check this by using the property Definition A.1 (2). Hence we define $\phi^+_\sigma(E) := \phi_1$, and $\phi^-_\sigma(E) := \phi_n$. We call the sequence the *Harder-Narashimhan filtration* (for short HN filtration) of $E$, and each $A_i$ a *semistable factor* of $E$. We also define $m_\sigma(E) = \sum_{A_i} m_{A_i}$ where $A_i$ runs through semistable factors of $E$. We call $m_\sigma(E)$ a *mass* of $E$.

(5) Let $I \subset \mathbb{R}$ be an interval. For $I$, we define $\mathcal{P}(I)$ as the extension closed additive full subcategory of $\mathcal{D}$ generated by $\mathcal{P}(\phi)$ $(\phi \in I)$. If $E \in \mathcal{P}(I)$, then $\phi^+(E)$ and $\phi^-(E) \in I$.

(6) A stability condition $\sigma$ is said to be *locally finite* if for all $\phi \in \mathbb{R}$, there is a positive number $\epsilon$ such that the quasi-abelian category $\mathcal{P}((\phi-\epsilon, \phi+\epsilon))$ is finite length, that is both increasing and decreasing sequences of subobjects of $A$ will terminate (See also §4 of [Bri07]). The property of local-finiteness guarantees the existence of Jordan-Hölder filtrations (for short JH filtrations), that is, for any $0 \neq A \in \mathcal{P}(\phi)$, there exists a sequence of distinguished triangles

$$\begin{tikzcd}
0 \arrow[rr] & & A_1 \arrow[rr] \arrow[dl] & & A_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & A_{n-1} \arrow[rr] & & A_n = A \arrow[dl] \\
& S_1 \arrow[ul, dashed, "{[1]}"] & & S_2 \arrow[ul, dashed, "{[1]}"] & & & & S_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

such that each $S_i$ is $\sigma$-stable with phase $\phi$. We call each $S_i$ a *stable factor* of $A$. We remark that JH filtrations may not be unique.

(7) The set of stability conditions on $\mathcal{D}$ has a metric defined by

$$d(\sigma, \tau) := \sup_{0 \neq E \in \mathcal{D}} \{|\phi^+_\sigma(E) - \phi^+_\tau(E)|, |\phi^-_\sigma(E) - \phi^-_\tau(E)|, |\log \frac{m_\sigma(E)}{m_\tau(E)}|\},$$

where $\sigma$ and $\tau$ are stability conditions on $\mathcal{D}$. The value $d(\sigma, \tau)$ may be $\infty$.

In general it is difficult to construct stability conditions on $\mathcal{D}$. However, by using Proposition A.4 (below), we can explicitly construct them in some cases. Before we state the proposition, we introduce the notion of a stability condition on abelian categories.

**Definition A.3.** Let $\mathcal{A}$ be an abelian category, and $Z : K(\mathcal{A}) \to \mathbb{C}$ a group homomorphism from the Grothendieck group $K(\mathcal{A})$ of $\mathcal{A}$ to $\mathbb{C}$, satisfying

$$Z(E) = m_E \exp(\sqrt{-1}\pi\phi_E) \text{ for } 0 \neq E \in \mathcal{A}, \text{ where } \phi_E \in (0, 1] \text{ and } m_E > 0.$$

We call $Z$ a *stability function* on $\mathcal{A}$. An object $E \in \mathcal{A}$ is called a *(semi)stable object* for $Z$ when, for any non-trivial subobjects $F$ of $E$, the following inequality holds:

$$\phi_F < \phi_E, (\phi_F \leq \phi_E).$$

If $Z$ has the following property, we call $Z$ a stability function equipped with the *Harder-Narashimhan (for short HN) property*:

$$0 \neq \forall E \in \mathcal{A}, \exists \text{ a filtration } 0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E \text{ such that}$$
$$A_i = E_i/E_{i-1} \text{ is semistable and } \phi_{A_1} > \cdots > \phi_{A_n}.$$

**Proposition A.4** ([Bri07, Proposition 5.3])**.** *Let $\mathcal{D}$ be a triangulated category. Then the following are equivalent:*

(1) *To give a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$.*

(2) *To give a pair* $(\mathcal{A}, Z_{\mathcal{A}})$ *consisting of the heart* $\mathcal{A}$ *of a bounded t-structure on* $\mathcal{D}$ *and a stability function* $Z_{\mathcal{A}}$ *on* $\mathcal{A}$ *which has the HN property.*

Let $\chi$ be the Euler paring on $K(\mathcal{D})$ and $\mathcal{N}(\mathcal{D})$ be the quotient of $K(\mathcal{D})$ by the numerical equivalence with respect to $\chi$. We call $\mathcal{N}(\mathcal{D})$ the *numerical Grothendieck group.*

**Definition A.5.** A stability condition $\sigma = (\mathcal{A}, Z)$ on $\mathcal{D}$ is said to be *numerical* if the central charge $Z$ factors through the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$.

In the following lemma, we introduce two actions of groups on $\mathrm{Stab}(X)$.

**Lemma A.6** ([Bri07, Lemma 8.2])**.** *Let* $\mathrm{Stab}(\mathcal{D})$ *be the space of stability condition on* $\mathcal{D}$*.* $\mathrm{Stab}(\mathcal{D})$ *carries a right action of* $\tilde{GL}^+(2, \mathbb{R})$*, and a left action of* $\mathrm{Aut}(\mathcal{D})$*. In addition, these two actions commute.*

**Remark A.7.** By the definition of the action of $\tilde{GL}^+(2, \mathbb{R})$, we can easily see that for any $\sigma \in \mathrm{Stab}(\mathcal{D})$ and any $\tilde{g} \in \tilde{GL}^+(2, \mathbb{R})$, $E \in \mathcal{D}$ is $\sigma$-(semi)stable if and only if $E$ is $\sigma \cdot \tilde{g}$-(semi)stable.

## Appendix B. Boundary of $U(X)$

In this appendix, we assume that $X$ is a projective K3 surface. We first recall a chamber structure theorem.

**Definition B.1.** Let $\mathcal{S}$ be a set of objects in $D(X)$. The set $\mathcal{S}$ has *bounded mass* in $\mathrm{Stab}^{\dagger}(X)$ if for some $\sigma$ we have

$$\sup\{m_{\sigma}(E) | E \in \mathcal{S}\} < \infty.$$

We remark that if $s(\sigma) = \sup\{m_{\sigma}(E) | E \in \mathcal{S}\} < \infty$ for some $\sigma \in \mathrm{Stab}^{\dagger}(X)$ then $s(\tau) < \infty$ for all $\tau \in \mathrm{Stab}^{\dagger}(X)$. The property bounded mass ensures the finiteness of Mukai vectors in $S$. Namely

**Proposition B.2** ([Bri08, Lemma 9.2])**.** *Suppose that the set* $\mathcal{S} \subset D(X)$ *has bounded mass in* $\mathrm{Stab}^{\dagger}(X)$*. Then the set* $\{v(E) | E \in \mathcal{S}\}$ *is finite.*

**Proposition B.3** ([Bri08, Propositions 9.3 and 9.4])**.** *Suppose* $\mathcal{S} \subset D(X)$ *has bounded mass.*

(1) *Take a compact subset* $B \subset \mathrm{Stab}^{\dagger}(X)$*. There exists a finite set* $\{W_{\gamma}\}_{\gamma \in \Gamma}$ *of real codimension* 1 *submanifolds such that* $\{W_{\gamma}\}_{\gamma \in \Gamma}$ *gives a chamber structure. Namely each connected component* $C \subset B \setminus \bigcup_{\gamma \in \Gamma} W_{\gamma}$ *satisfies the following: If* $E \in \mathcal{S}$ *is* $\sigma_0$*-semistable for some* $\sigma_0 \in C$ *then* $E$ *is* $\sigma$*-semistable for all* $\sigma \in C$*.*
(2) *In addition to* (1) *suppose that the all Mukai vectors* $v(E)$ *of* $E \in \mathcal{S}$ *is primitive. The set* $\{\sigma \in \mathrm{Stab}^{\dagger}(X) | \forall E \in \mathcal{S}$ *is* $\sigma$*-stable*$\}$ *is open in* $\mathrm{Stab}^{\dagger}(X)$*.*

*Proof.* For the convenience of readers we recall the construction of walls $\{W_\gamma\}_{\gamma\in\Gamma}$.

Let $\mathcal{T}$ be the set of objects

$$\mathcal{T} = \{A \in D(X) | \exists \sigma \in B, \exists E \in \mathcal{S} \text{ such that } m_\sigma(A) \leq m_\sigma(E)\}.$$

Let $\Gamma$ be the pair of Mukai vectors in the set $\mathcal{T}$ which are not proportional:

$$\Gamma = \{(v,w) | v = v(A), w = v(B) (\exists A, B \in \mathcal{T}) \text{ with } v \notin \mathbb{R}w\}.$$

Since $\mathcal{T}$ has bounded mass, $\Gamma$ is finite by Proposition B.2. For each $\gamma = (v,w) \in \Gamma$, the wall $W_\gamma$ is given by

$$W_\gamma = \{\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}^\dagger(X) | Z(v)/Z(w) \in \mathbb{R}_{>0}\}.$$

□

By the definition of $U(X)$, we see that the closure of $U(X)$ is written as

$$\bar{U}(X) = \{\sigma \in \mathrm{Stab}^\dagger(X) | \mathcal{O}_x \text{ is } \sigma\text{-semistable for all } x \in X \text{ in the same phase}\}.$$

Since the set $\{\mathcal{O}_x\}_{x\in X}$ has bounded mass, $\partial U(X) = \bar{U}(X)\backslash U(X)$ is a locally finite union of real codimension 1 submanifolds of $\mathrm{Stab}^\dagger(X)$ by Proposition B.3. Thus we can put

$$\partial U(X) = \bigcup_{\gamma\in\Gamma} W_\gamma.$$

**Definition B.4.** For $\sigma \in \partial U(X)$, $\sigma$ is said to be *in a general position* if there exists only one $\gamma \in \Gamma$ such that $\sigma \in W_\gamma$.

Since $\mathcal{O}_x$ is $\sigma$-semistable for $\sigma \in \partial U(X)$ there exists a Jordan-Hölder filtration of $\mathcal{O}_x$. For general $\sigma \in \partial U(X)$, Bridgeland determined it.

**Theorem B.5** ([Bri08, Theorem 12.1])**.** *Let $\sigma \in \partial U(X)$ be general. Then exactly one of the following holds:*

($A^+$) *There is a spherical locally free sheaf $A$ such that both $A$ and $T_A(\mathcal{O}_x)$ are stable factors of $\mathcal{O}_x$ for any $x \in X$, where $T_A$ is the spherical twist by $A$. Moreover a JH filtration of $\mathcal{O}_x$ is given by*

$$A^{\oplus \operatorname{rank} A} \longrightarrow \mathcal{O}_x \longrightarrow T_A(\mathcal{O}_x) \longrightarrow A^{\oplus \operatorname{rank} A}[1].$$

*In particular $\mathcal{O}_x$ is properly $\sigma$-semistable*[2] *for all $x \in X$ and $A$ does not depend on $x \in X$.*

($A^-$) *There is a spherical locally free sheaf $A$ such that both $A$ and $T_A^{-1}(\mathcal{O}_x)$ are stable factors of $\mathcal{O}_x$ for any $x \in X$, where $T_A$ is the spherical twist by $A$. Moreover a JH filtration of $\mathcal{O}_x$ is given by*

$$T_A^{-1}(\mathcal{O}_x) \longrightarrow \mathcal{O}_x \longrightarrow A^{\oplus \operatorname{rank} A}[2] \longrightarrow T_A^{-1}(\mathcal{O}_x)[1].$$

*In particular $\mathcal{O}_x$ is properly $\sigma$-semistable for all $x \in X$ and $A$ does not depend on $x \in X$.*

[2]Namely $\mathcal{O}_x$ is not $\sigma$-stable but $\sigma$-semistable.

($C_k$) *There are a* $(-2)$*-curve* $C$ *and an integer* $k$ *such that* $\mathcal{O}_x$ *is* $\sigma$*-stable if* $x \notin C$ *and* $\mathcal{O}_x$ *is properly* $\sigma$*-semistable if* $x \in C$*. Moreover a JH filtration of* $\mathcal{O}_x$ *for* $x \in C$ *is given by*

$$\mathcal{O}_C(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_C(k)[1] \longrightarrow \mathcal{O}_C(k+1)[1].$$

## References


[ABL07] Daniel Arcara, Aaron Bertram and Max Lieblich, Bridgeland-stable moduli spaces for K-trivial surfaces. preprint, arXiv:0708.2247.

[BMT11] Arend Bayer, Emanuele Macrì and Yukinobu Toda, Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. preprint, arXiv:1103.5010.

[Bri07] Tom Bridgeland, Stability conditions on triangulated categories. Ann. of Math. **166** (2007), 317–345.

[Bri08] Tom Bridgeland, Stability conditions on K3 surfaces. Duke Math. Journal **141** (2008), 241–291.

[Har10] Heinrich Hartmann, Cusps of the Kähler moduli space and stability conditions on K3 surfaces. preprint, arXiv:1012.3121.

[Huy07] Daniel Huybrechts, Fourier-Mukai transformations in Algebraic Geometry. Oxford Mathematical Monographs, 2007.

[Huy08] Daniel Huybrechts, Derived and abelian equivalence of K3 surfaces. Journal of Algebraic Geometry **17** (2008), 375–400.

[HL97] Daniel Huybrechts and Manfred Lehn, The geometry of moduli spaces of sheaves. Aspects of Mathematics, 1997.

[HMS09] Daniel Huybrechts, Emanuele Macrì and Paolo Stellari, Derived equivalences of K3 surfaces and orientation. Duke Math. Journal **149** (2009), 461–507.

[Kaw10] Kotaro Kawatani, Stability conditions and $\mu$-stable sheaves on K3 surfaces with Picard rank one. preprint, arXiv:1005.3877 to appar in Osaka Journal Math.

[Kaw11] Kotaro Kawatani, Stability of Gieseker stable sheaves on K3 surfaces in the sense of Bridgeland and some applications. preprint, arXiv:1103.3921 (2011).

[Kul90] Sergej A. Kuleshov, An existence theorem for exceptional bundles on K3 surfaces. Math. USSR Izv., **34** (1990) 373–388.

[Mac12] Antony Maciocia, Computing the walls associated to Bridgeland stability conditions on projective surfaces. preprint, arXiv:1202.4587 (2012).

[Mac07] Emanuele Macrì, Stability conditions on curves. Math. Res. Letters, **14** (2007) 657–672.

[Muk87] Shigeru Mukai, On the moduli spaces of bundles on K3 surfaces, I. In:Vector Bundles on Algebraic Varieties, Oxford Univ. Press, (1987) 341–413.

[Oka06] So Okada, Stability manifold of $\mathbb{P}^1$. Journal of Algebraic Geometry **15** (2006), 487–505.

[Orl97] Dmitri Orlov, Equivalences of derived categories and K3 surfaecs. J. Math. Sci. **84** (1997), 1361–1381.

[ST01] Paul Seidel and Richard Thomas, Braid group actions on derived categories of coherent sheaves, Duke Math. Journal, **108** (2001) 37–108.

[Woo12] Jon Woolf, Some metric properties of spaces of stability conditions. preprint, arXiv:1108.2668 (2011).



Department of Mathematics, Kyoto University, Oiwakecho, Kitashirakawa, Kyoto, Japan

*E-mail address*: `kawatani@math.kyoto-u.ac.jp`